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Quadratic volume-preserving maps

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Abstract. We study quadratic, volume-preserving diffeomorphisms whose inverse is also quadratic. Such maps generalize the Hénon area-preserving map and the family of symplectic quadratic maps studied by Moser. In particular, we investigate a family of quadratic volume-preserving maps in three-space for which we find a normal form and study invariant sets. We also give an alternative proof of a theorem by Moser classifying quadratic symplectic maps.

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1. Introduction

The study of the dynamics of polynomial mappings has a long history both in applied and pure dynamics. For example, such mappings provide simple models of the motion of charged particles through magnetic lenses and are often used in the study of particle accelerators [1]. The simplest nonlinear systems are given by quadratic maps; the quadratic, area-preserving map, introduced by Hénon [2], is one of the simplest models of chaotic dynamics.

Hénon's study can be generalized in several directions. For example, Moser [3] studied the class of quadratic, symplectic maps, obtaining a useful decomposition and normal form. Here we do the same for a more general class of quadratic, orientation preserving volume-preserving maps, with one caveat as we discuss below.

Just as symplectic maps arise as Poincaré maps of Hamiltonian flows, volume-preserving transport is based on a partition of phase space into regions separated by partial barriers that restrict the motion in some way [17]. For example, in two dimensions a partition is formed from intersecting stable and unstable manifolds of a saddle periodic orbit. In higher dimensions an analogous construction requires the existence of codimension-one manifolds that separate the space [18]. In most cases it is difficult to find a dynamically natural construction of such manifolds; however, such manifolds do appear in volume-preserving maps, and this leads easily to the construction of partial barriers.

The computation and effective visualization of invariant manifolds in higher-dimensional maps is itself an interesting problem [19]. In this paper we will study the intersections of the two-dimensional stable and unstable manifolds in $\mathbb{R}^3$.

2. Quadratic shears

In this section we will study maps of the form

$$x \mapsto x + \tfrac{1}{2}Q.x/$$

where Q is a vector of quadratic polynomials. Throughout this paper we will write vectors of quadratic polynomials using the form $Q.x/ = M.x$

This implies that the characteristic polynomial of $M(x)$ is $(\;-\;)^n$ and therefore $[M(x)]^n = 0$. □

At this point, we restrict ourselves to the case of quadratic maps in standard form

Definition 2.2. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be given, in standard form, by $f.x/ = x + \frac{1}{2}M.x/x$. If f satisfies any of the conditions of lemma 2.2, we will say that f is a quadratic shear.

A simple family of quadratic shears is determined by any vector $v \in \mathbb{R}^n$ and a symmetric matrix P such that $Pv = 0$ according to $M.x/y = .x^T Py/v$, for then

$$M.x/^2 x = .x^T Pv/.x^T Px/v = 0:$$

We will see that, at least in the case $n = 3$, this is the most general quadratic shear. Moser's normal form for symplectic, quadratic maps [3] shows that the higher-dimensional case is not quite this simple. From now on, we will concentrate on the special case $n = 3$.

Theorem 2.1. *A function $f : \mathbb{R}^3 \to \mathbb{R}^3$ is a quadratic shear in $\mathbb{R}^3$ if and only if there is a vector $v \in \mathbb{R}^3$ and a 3×3 symmetric matrix P such that $Pv = 0$ and*

$$f.x/ = x + \frac{1}{2}.x^T Px/v:$$

Proof. Since f is a bijection, we can define a new function $g : S^2 \to S^2$ on the unit two-dimensional sphere $S^2 \subset \mathbb{R}^3$, in the following way.

Let $M_1 = M.e_1/; M_2 = M.e_2/$ and $M_3 = M.e_3/$ where $e_1 = .1; 0; 0/; e_2 = .0; 1; 0/$ and $e_3 = .0; 0; 1/$. It is easy to see that

$$M_1 = \begin{pmatrix} 0 & & 0 \\ 0 & 0 & 0 \\ 0 & & 0 \end{pmatrix};$$

i.e. $S.x/ = x + \frac{1}{2}M.x/x$, where $M.x/$ is linear in x and satisfies the symmetry property $M.x/y = M.y/x$. Then S is symplectic providing

$$.I + M.x//^T J .I + M.x// = J:$$

Homogeneity of $M.x/$ implies that

$$M.x/^T J = J$$

4. Normal form in $\mathbb{R}^3$

in this coordinate system

$$U^{-1}LUe_1 = U^{-1}L^2v = e_2 + \quad e_1$$
$$U^{-1}LUe_2 = U^{-1}.L^3v - \quad L^2v/$$
$$U^{-1}LUe_3 = U^{-1}Lv = e_1:$$

The second equation can be simplified by noting that the characteristic equation for the matrix L is satisfied by L itself, and so $L^3 - \quad L^2 + \quad L - I = 0$, where $\quad = \mathrm{Tr}.L/$ and $\quad = \mathrm{Tr}_2.L/$, the ‘second trace’ of the matrix L, thus we obtain $U^{-1}LUe_2 = U^{-1}.I - \quad L/v = e_3 - \quad e_1$. Thus we obtain

$$U^{-1}LU = \begin{pmatrix} & - & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}:$$

Upon reverting to $.x; y; z/$ as the names for the coordinates we obtain

$$U^{-1}f.U.x// = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \begin{pmatrix} x - \quad y + z + Q.x; y/ \\ x \\ y \end{pmatrix}:$$

To simplify this map further, we can conjugate, using the translation

$$.x; y; z/ \mapsto .x; y + y_0; z + y_0 + z_0/;$$

to a map with $x_0 = \quad$, $y_0 = 0$ and $z_0 = 0$. This is the promised form.

For the second case, assume that $L^2v = \quad Lv - \quad v$, for some nonzero and . This implies that the characteristic polynomial for L factors as $.L - 1= \quad I/.L^2 - \quad L + \quad I/ = 0$, and therefore, since L is nondegenerate, there exists a vector $w \in Z.v; L/$ such that $Lw = \frac{1}{\ } w$. We define the following change of coordinates.

$$U^{-1}v = e_2 \qquad Ue_2 = v \tag{7}$$
$$U^{-1}Lv = e_1 \qquad Ue_1 = Lv \tag{8}$$
$$U^{-1}w = e_3 \qquad Ue_3 = w \tag{9}$$

As before, we note that in the new coordinates the quadratic term satisfies $\tilde{Q}.e_2; \quad / = 0$, so in the new coordinates the quadratic terms depend only on the first and third components. Moreover in this coordinate system we obtain

$$U^{-1}LU = \begin{pmatrix} & 1 & 0 \\ - & 0 & 0 \\ 0 & 0 & \frac{1}{\ } \end{pmatrix}:$$

This implies the form for the second case.

For the third case, assume that $Lv = \quad v$

5. Dynamics

The dynamics of the second and third cases of theorem 4.1 are essentially trivial. In case (ii), the z dynamics decouples from the $.x; y/$ dynamics. There are four special cases.

- $| \ | \neq 1$. The plane $z = \ z_0 = . \ - 1/$ is invariant, and is either a global attractor ($| \ | > 1$) or repellor ($| \ | < 1$). On the plane the dynamics is linear.
- $= 1$, $z_0 \neq 0$. All orbits are unbounded.
- $= 1$, $z_0 = 0$. Every plane $z = c$ is invariant, and the dynamics reduces to a two-dimensional, area-preserving Hénon map on each plane.
- $= -1$. Each plane $z = c$ is fixed under f^2. Restricted to a plane, f^2 is the composition of two orientation-reversing Hénon maps.

For case (iii) the $.y; z/$ dynamics is linear and decouples from the x dynamics. Generically, there is an invariant line on which the dynamics is affine. The invariant line can have any stability type.

5.1. *Generic case*

Equation (6) is the only nontrivial case. In general this map has six parameters, one from the shift, two from the linear matrix (the two coefficients of its characteristic polynomial) and the three coefficients of Q. However, generically, two of these parameters can be eliminated.

Write the quadratic form as $Q.x; y/ = ax^2 + bxy + cy^2$. Generically $a + b + c \neq 0$ and we can we can apply a scaling transformation to set $a + b + c = 1$. Similarly if $b + 2c \neq 0$ the parameter can be eliminated using the diagonal translation

$$.x; y; z/ \mapsto .x + \gamma; y + \gamma; z + \gamma/; \qquad \gamma = \ = .b + 2c/:$$

In this way, we obtain the final, generic form

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \ + \ x + z + ax^2 + bxy + cy^2 \\ x \\ y \end{pmatrix} \qquad a + b + c = 1: \tag{10}$$

There are four parameters in the system. Even if $a + b + c = 0$ and/or $b + 2c = 0$, then other normalizations can be chosen to eliminate two of the parameters in (6). We will not study these special cases.

5.2. *Periodic orbits*

Generically we can assume that $a + b + c = 1$ for the quadratic form in (6). The map (6) has at most two fixed points

$$x = y = z = x_{\pm} = \tfrac{1}{2}\left(- \ + \ \pm \sqrt{. \ - \ /^2 - 4 \ }\right) \tag{11}$$

born in a saddle-node bifurcation at $. \ - \ /^2 - 4 \ = 0$. The characteristic polynomial of the linearized map at the fixed points is

$$^3 - t \ ^2 + s \ - 1 = 0$$

where the trace t and second trace s are

$$t_{\pm} = \ + .2a + b/x_{\pm}$$
$$s_{\pm}$$

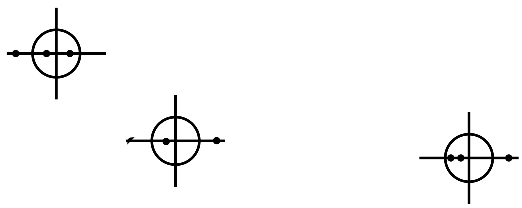


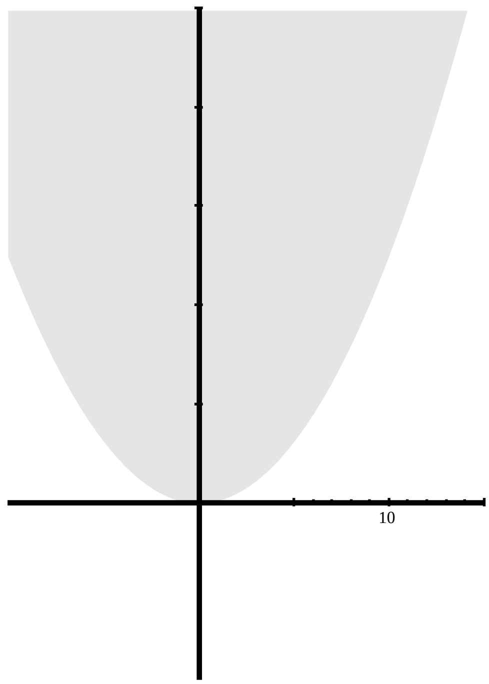
Figure 2. Two-dimensional stable and unstable manifolds for the parameters $a = c = 0{:}5$, $b = 0{:}0$, $\ = 0{:}0$, $\ = -0{:}3$ and $\ = 0{:}0$.

same side of the period-doubling line, so that one is type *A* and the other type *B*; however, when this parameter is large enough they can be on opposite sides, and therefore of the same *type*. This is determined by the sign of

$$s_\pm + t_\pm + 2 = 2 + \quad + \quad + 2.a - c/x_\pm{:}$$

When $a = c$, we have $t_\pm = s_\mp$ so that the eigenvalues of the two fixed points are reciprocal (see section 5.3 for the explanation of this).

Remember that, generically and without loss of generality, we can assume that $\ = 0$. Therefore, we can plot stability diagrams for different values of and . The stability diagram in the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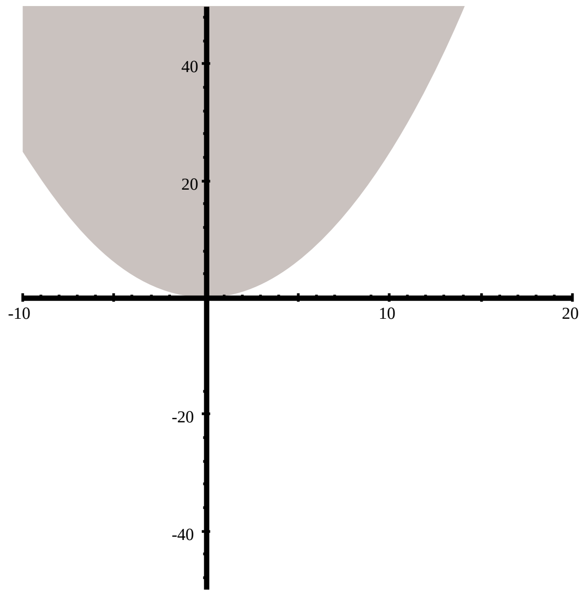
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-20
-40

We conclude that $h(x_+) = x_-$. This implies that $t_\pm = s_\mp$, which gives $a = c$, or $Q(x, y) = Q(y, x)$.

The other direction is proved by a simple computation, as described above. □

Reversibility simplifies finding orbits of a map. Orbits that are invariant under h are called symmetric, and as is easy to see, they must have points on the fixed set $\mathrm{Fix}(h) = \{x \in \mathbb{R}^3 : h(x) = x\}$. In our case this is the line $x + z = - \ $, $y = - \ /2$, so a numerical algorithm for finding for symmetric orbits involves a one-dimensional search.

Similarly, if the stable manifold of one of the fixed points intersects $\mathrm{Fix}(h)$, then the intersection point is on a heteroclinic orbit, for suppose $x \in \mathrm{Fix}(h) \cap W^s(x_+)$, then $x \in W^u(x_+)$, because $h(f^n(x)) = f^{-n}(h(x)) = f^{-n}(x)$, so

$$\lim_{n\to\infty} f^n(x) = x_+ \Rightarrow \lim_{n\to\infty} h(f^n(x)) = \lim_{n\to\infty} f^{-n}(x) = h(x_+) = x_-.$$

Furthermore, suppose the stable manifold is two dimensional, and has the normal vector $\hat{n}$ at a point on $\mathrm{Fix}(h)$, then $Dh(x)\hat{n}$ is the normal to the unstable manifold at this point. This implies that the curve of heteroclinic orbits is tangent to the direction $\hat{n} \times Dh(x)\hat{n}$.

5.4. Bounded orbits

For the Hénon map, it is well known that the set of bounded orbits is contained in a square. For the volume-preserving case, we will show that an analogue of this result also holds providing the quadratic form Q is positive definite.

Theorem 5.1. *If Q is positive definite then there is a $\ > 0$ such that all bounded orbits are contained in the cube $\{(x, y, z) : |x| \leqslant \ , |y| \leqslant \ , |z| \leqslant \ \}$. Moreover, points outside the cube go to infinity along the $+x$ axis as $t \to +\infty$ or the z-axis as $t \to -\infty$.*

Proof. We start by writing the map in third-difference form as

$$x_{t+1} = \ + \ x_t - \ x_{t-1} + x_{t-2} + Q(x_t, x_{t-1}).$$

Recall that a quadratic form $Q(x, y) = ax^2 + bxy + cy^2$

is monotone increasing. In fact, this sequence is unbounded; otherwise it would have a limit $x_t \to x^* > {}_1$, and this point would have to be a fixed point $x = y = z = x^*$

the components are negative, and so the orbit moves to infinity in the negative octant as $t \to -\infty$. Once all components are negative, we have

$$\frac{z_{t-1}}{z_t} = x_{t-3}$$

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