

Breakup of invariant tori for the four-dimensional semi-standard map

Erik M. Boltt and James D. Meiss

Program in Applied Mathematics, Box 576, University of Colorado, Boulder, CO 80309, USA

Received 10 October 1992
Accepted 25 January 1993

We compute the domain of existence of two-dimensional invariant tori with fixed frequency vectors for a class of four-dimensional semi-standard maps.

Explicit bounds on the domain for fixed k are obtained. Numerical results show that quadratic irrationals can be more stable than cubic irrationals. The results are compared with those of the corresponding three-dimensional case.

Keywords: Invariant tori; stability; dynamics; bifurcation; numerical analysis; introduction

MSC: 37J45; 37J46; 37J47; 37J48; 37J50; 37J51; 37J52; 37J53; 37J54; 37J55; 37J56; 37J57; 37J58; 37J59; 37J60; 37J61; 37J62; 37J63; 37J64; 37J65; 37J66; 37J67; 37J68; 37J69; 37J70; 37J71; 37J72; 37J73; 37J74; 37J75; 37J76; 37J77; 37J78; 37J79; 37J80; 37J81; 37J82; 37J83; 37J84; 37J85; 37J86; 37J87; 37J88; 37J89; 37J90; 37J91; 37J92; 37J93; 37J94; 37J95; 37J96; 37J97; 37J98; 37J99; 37K01; 37K02; 37K03; 37K04; 37K05; 37K06; 37K07; 37K08; 37K09; 37K10; 37K11; 37K12; 37K13; 37K14; 37K15; 37K16; 37K17; 37K18; 37K19; 37K20; 37K21; 37K22; 37K23; 37K24; 37K25; 37K26; 37K27; 37K28; 37K29; 37K30; 37K31; 37K32; 37K33; 37K34; 37K35; 37K36; 37K37; 37K38; 37K39; 37K40; 37K41; 37K42; 37K43; 37K44; 37K45; 37K46; 37K47; 37K48; 37K49; 37K50; 37K51; 37K52; 37K53; 37K54; 37K55; 37K56; 37K57; 37K58; 37K59; 37K60; 37K61; 37K62; 37K63; 37K64; 37K65; 37K66; 37K67; 37K68; 37K69; 37K70; 37K71; 37K72; 37K73; 37K74; 37K75; 37K76; 37K77; 37K78; 37K79; 37K80; 37K81; 37K82; 37K83; 37K84; 37K85; 37K86; 37K87; 37K88; 37K89; 37K90; 37K91; 37K92; 37K93; 37K94; 37K95; 37K96; 37K97; 37K98; 37K99

dynamics, an understanding of stability is also of

it contains a practical stability since orbits ap-

of tori is the main theorem which asserts that

ORBITS ARE STABLE TO SOME EXTENT THIS IS

incommensurate frequency vectors (they satisfy

equivalently one can consider a sequence of

of the form (k, ω) with $k \in \mathbb{Z}^2$ and $\omega \in \mathbb{R}^2$

of the form (k, ω) with $k \in \mathbb{Z}^2$ and $\omega \in \mathbb{R}^2$

the perturbation size for the destruction of tori

stability of a sequence of periodic orbits whose

the theorem guarantees the survival of any
Dionthantine torus for any small enough pertur-

parameters at which invariant circles are de-
stroyed and can be made rigorous [16]. The

some cases using interval arithmetic) of the con-

Though many have attempted to generalize

more than two degrees of freedom, or equivalently, symplectic maps of four or more dimensions, there has been limited success in determining

approximations to invariant tori have been obtained [14, 25], and computations reveal that the

It is worth noting that the method

is not applicable to the case of the

Since is number theoretic, there is no systematic

theory to simultaneous approximation of several

of several variables (in the case of the

though if the invariance is furthermore

because their continued fraction expansions are eventually periodic (these give rise to self-similar structures). Finally, the most robust tori appear to correspond to the class of quadratic irrationals. Because of the noble numbers, these have a

Roughly speaking, the explanation for this is that

There has been some speculation that for four-

therefore cubic [11, 12]. However, even in this

self-similar behavior near breakup [21], and there is no evidence that cubic irrationals are more robust than others.

complex, symplectic map corresponding to the coupling of two semi-standard maps, as intro-

of the form $F_1 \circ F_2 \circ F_1^{-1} \circ F_2^{-1}$

of the form $F_1 \circ F_2 \circ F_1^{-1} \circ F_2^{-1}$

Fourier series for the semi-standard map, we are

holomorphic functions of several complex vari-

of several variables. One needs to check the convergence of the Fourier series has a particular

example, frequency vector

2. Coupling of two semi-standard maps

The semi-standard map appearing here was

numerically simpler model than the standard

the semi-standard map takes $\{x_{t-1}, x_t\} \mapsto$

$$x_t \equiv x_{t-1} + \epsilon \frac{\partial \mathcal{H}}{\partial x_{t-1}} \quad (1)$$

with $\mathcal{H} \equiv x_{t-1}^2 - 2x_t + x_{t-1}^2 \equiv \epsilon a e^{i\theta}$; thus $x(\theta)$ is periodic

this is a map on the notation δ is reminiscent of the second derivative operator.

In this paper we study a four-dimensional map where $\chi(\theta)$ is doubly 2π periodic. If we suppose

THESE MAPS IS

$$x_{t+1} = x_t + \epsilon \frac{\partial \mathcal{H}}{\partial x_t} \quad (2)$$

eq. (3) into eq. (2) yields the Percival equation

$$F(x) \equiv 1 - \frac{\epsilon}{a_2} \frac{\partial \mathcal{H}}{\partial x} = 0 \quad (3)$$

There are three parameters, the strength of the coupling (a_1, a_2) and ϵ , the strength of the coupling of the two maps. Eq. (2) is symplectic since F is the gradient of a scalar potential (see for example [14]).

We are looking for solutions x of eq. (2) that are analytic and periodic on the torus to be analytically conjugate to a uniform rotation

$$x_t \longrightarrow x_{t+1}$$

value and exists a maximum



thus, for a given ω , an invariant torus for eq. (2) is given by

$$x_t = x(\theta + 2\pi\omega t), \quad (5)$$

$$0 = x(\theta) = x(\theta + 2\pi\omega) - 2x(\theta) + x(\theta - 2\pi\omega)$$

equations determining the Fourier coefficients χ ; these will be obtained in section 4.

3. Incommensurate frequencies

[10, 21, 25]. In particular, rather sophisticated

for which there is an analytic invariant circle with frequency ω . Here a^{ss} , the critical function, is zero for every rational ω and a maximum for

The critical function appears to have a local maximum at each of the noble frequencies. Those equivalent to γ under a modular transformation,

or equivalently which have a continued fraction

K

These results also apply to the semi-Froeshlé map when $\epsilon = 0$. Thus an invariant torus of

If $d = 1$ then K can be replaced by $1/\sqrt{5}$ but nothing smaller.

KAM theory implies that for sufficiently small

Dionhantine are those constructed from alge-

energy vector satisfies a Diophantine condition.

with ω_μ consists of those $\omega \in \mathbb{R}^d$ for which there exists a $C > 0$ such that for all $(p, q) \in \mathbb{Z}^{d+1}$

Recall that an algebraic field generated by $\xi \in \mathbb{R}$ of degree n is defined as the set of numbers of the form

$$|p \cdot \omega - q| \geq \frac{C}{\|p\|^\mu}, \quad (11)$$

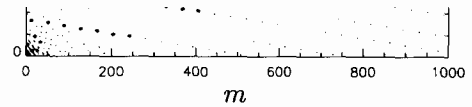
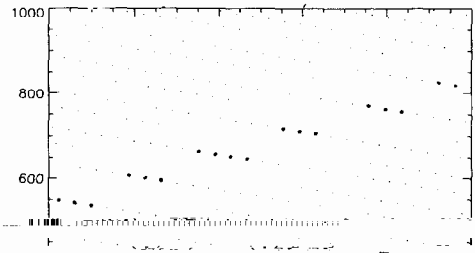
$$R(\xi) = \frac{P(\xi)}{Q(\xi)}$$

$$\sigma \in \sqrt{2} = [2, 2, 2, 2, \dots] \equiv [2^\infty],$$

$$\xi \equiv \frac{1 + \sqrt{2}}{5 + 4\sqrt{2}} = [0, 4, 2^\infty]. \tag{13}$$

The expressions on the right hand sides above give the continued fraction expansions. Setting

easy to see, since $\xi^4 - 14\xi^2 + 9 = 0$ and any polynomial in ξ has the form $P(\xi) = a + b\sqrt{2} + c\sqrt{5} + d\sqrt{10}$ for $a, b, c, d \in \mathbb{Z}$. Thus γ, σ , and ξ



$$\tau^3 = \tau + 1,$$

$$\tau \approx$$

$$[1, 2, 10, 1, 1, 2, 2, 2, 4, 2, 141, 89, 2]$$

$$[5, 1, 2, 8, 2, 1, 1, 3, 1, \dots]. \tag{14}$$

This so called "spiral mean" frequency was in-

elements appear unbounded [22]. None-the-less,

τ is the smallest of the "PV numbers", which

terms of the small denominators

$$D_n \equiv 4 \sin^2(\pi n \tau) \approx 1.324717957244746025960908854 \tag{15}$$

pect the Fourier coefficients to have similar iso-

lately not the case.

where $\chi(\theta) = \chi(\theta) - \theta$.

and

where $\chi(\theta) = \chi(\theta) - \theta$.

It is convenient to consider $u \in \mathbb{C}^2$

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \quad (16)$$

The advantage of this definition is that the

our expansions. Further, using eq. (7) we define

$$g(u) = i[\chi(\theta) - \theta] = i\chi(\theta). \quad (17)$$

be needed in the Fourier expansion of $\chi(\theta)$. $g(u)$ has a cyclic expansion

$$g(u) = \sum_{m \in \mathbb{Z}^2} h_m u^m = \sum_{m \in \mathbb{Z}^2} (b_m^{(1)}) u^m$$

where we use standard multi-index notation for the vector exponentiation: while $u \in \mathbb{C}^2$ and $n \in \mathbb{N}^2$, $u^n = u_1^{n_1} u_2^{n_2} \in \mathbb{C}$. In addition to the expansion

we need the expansions of its complex conjugate as well:

$$e^{g_i(u)} = \sum_n c_n^{(i)} u^n, \quad (19)$$

where $g_i(u) = -i\chi(\theta)$

$$-k(u_1 u_2 e^{g_1(u)} e^{g_2(u)}). \quad (20)$$

where $\chi(\theta) = \chi(\theta) - \theta$.

and

where $\chi(\theta) = \chi(\theta) - \theta$.

It is convenient to consider $u \in \mathbb{C}^2$

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \quad (21)$$

The advantage of this definition is that the

our expansions. Further, using eq. (7) we define

$$g(u) = i[\chi(\theta) - \theta] = i\chi(\theta). \quad (17)$$

be needed in the Fourier expansion of $\chi(\theta)$. $g(u)$ has a cyclic expansion

$$g(u) = \sum_{m \in \mathbb{Z}^2} h_m u^m = \sum_{m \in \mathbb{Z}^2} (b_m^{(1)}) u^m$$

where we use standard multi-index notation for the vector exponentiation: while $u \in \mathbb{C}^2$ and $n \in \mathbb{N}^2$, $u^n = u_1^{n_1} u_2^{n_2} \in \mathbb{C}$. In addition to the expansion

we need the expansions of its complex conjugate as well:

$$e^{g_i(u)} = \sum_n c_n^{(i)} u^n, \quad (19)$$

where $g_i(u) = -i\chi(\theta)$

$$-k(u_1 u_2 e^{g_1(u)} e^{g_2(u)}). \quad (20)$$

If ω is incommensurate, then D_n is nonzero, so

define the partial order $<$ on integer vectors by

A simple derivative identity allows us to find

$$\frac{d}{du} e^{g_i(u)} = \left(\frac{d}{du} g_i(u) \right) e^{g_i(u)} \quad (22)$$

$$c_n^{(i)} = \sum_{m \neq (0,0)}^n \frac{m}{n} h_m^{(i)} c_{n-m}^{(i)} \quad (23)$$

Note that eq. (23) allows the two forms, $j=1$ or 2 , for n off the axis (these are equivalent), but for n on the axis, only one is valid because of a

examination of the mapping eq. (24) yields in addition the interior of the set of points for which it converges absolutely. A *polydisk* is the appropriate

$$b^{(2)} = 0 \quad b^{(1)} = \prod$$

$$(26) \quad 1 \quad d \} \text{ A Reinhardt domain is a domain } R$$

$D^{(1)}$ and $D^{(2)}$ are identical to those for the respectively.

Such a domain is *complete* if for every $z \in R$ the polydisk with smaller radii. Finally a domain D is

the domain of convergence of $\phi(u)$ is the each component's series.

is a convex subset of \mathbb{R}^d .

terms at a point z then it converges absolutely to a holomorphic function. The domain of convergence, D , of f is the interior of the set for which $\{b_n z^n\}$ is bounded. Furthermore, D is a loc-

ly, if $\{b_n z^n\}$ is unbounded then there is an order-

$$S = \sum b_m z^m, \quad (28)$$

Its most universal aspect is that the domain of

in more detail. Suppose $z, x \in \mathbb{C}^d \cap D$. Then for $\alpha + \beta = 1$ let u be any point in \mathbb{C}^{d*} such that

similar to the series obtained in the previous projection onto the radius space is denoted H :

where r_i and s_i are the radii of z and x , respec-

$x \cdot B \equiv \sin(|b_n z^n| - |b_n x^n|)$ exists and

straightforward since the series eq. (18) has the domain $\text{Re}(s) > 1$; it yields the asymptotic result

Corollary 1. For fixed k defined by eq. (21), an analytic invariant torus with Diophantine fre-

quency $\omega = (\omega_1, \omega_2)$ and action $I = (I_1, I_2)$ is bounded by the function $r_1(r_2)$ or $r_2(r_1)$.

The intersection of these domains is possible with reasonable accuracy if the series must be bounded.

6. Numerical results

They are real and positive for $n = 0$. The next

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{m+n+1} \approx 2.303$$

converges rapidly in the region $\text{Re}(s) > 1$.

the i th component is

$$\log(r_1^{(i)}(s)) = - \lim_{n \rightarrow \infty} \frac{\sum_{k=0}^n \log(r_1^{(i)}(s))}{n} \quad (36)$$

for each fixed s . Since the domain of conver-

gence is a small region, we

lead us to choose $N = 255$ as our matrix di-

agonalization. The efficiency of this method depends upon

of the Fourier coefficients. We first consider

$\omega = (\omega_1, \omega_2)$ where the components were defined by eqs. (10) and (15). Fig. 2 is a logarithmic

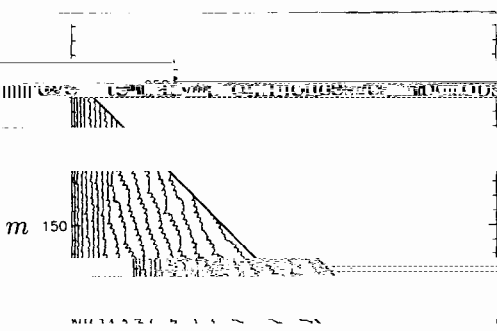
plot of $\log |c_m|$ versus m . The horizontal values in

the recursion algorithm, which shows that c_m is a

jump. This can be seen in the contour plot as a

change in slope for greater m are influenced by this

jump and dominates the entire plot. In other



extremely important, and primary, secondary, and
 that the Fourier coefficients have an extremely

coupling between the frequencies ω appears to
 play the dominant role. Resonances gain and
 lose asymptotic dominance relative to one another.

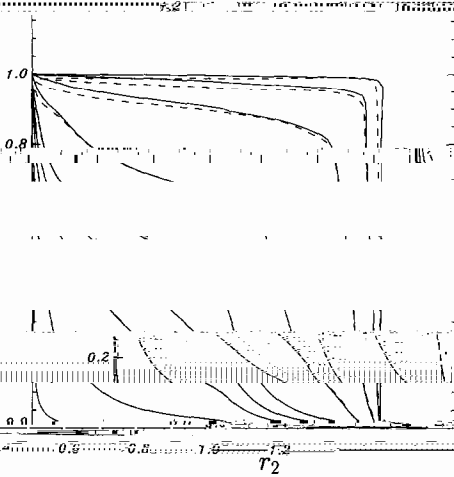


Fig. 4. Boundary of the domain of convergence for $\omega = \gamma$.

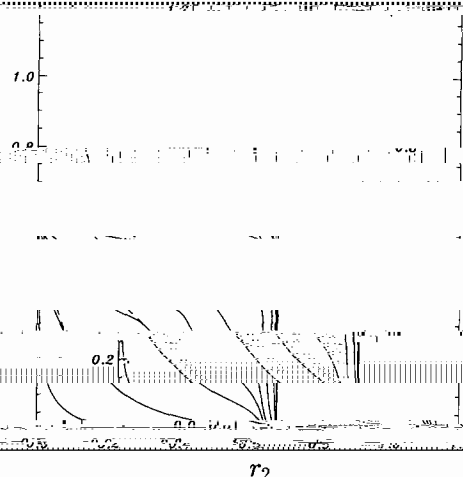


Fig. 6. Same as fig. 4 with $\omega = (\tau, \tau^2)$.

corresponds to the frequency $\omega = \gamma$, and the horizontal axis

frequencies. Note that the curves in figs. 4 and 6

and, for example in fig. 4, the intersection with the r_2 axis occurs near 0.985, while table 1 implies that the correct value is 0.966. This overestimate is due to the fact that we compute the coefficients only out to the 255th Fourier

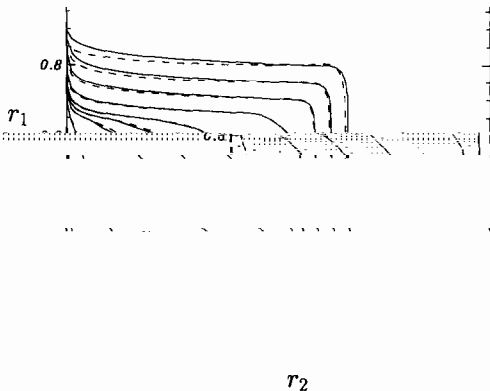


Fig. 5. Same as fig. 4 with $\omega = (\gamma, \gamma^2)$.

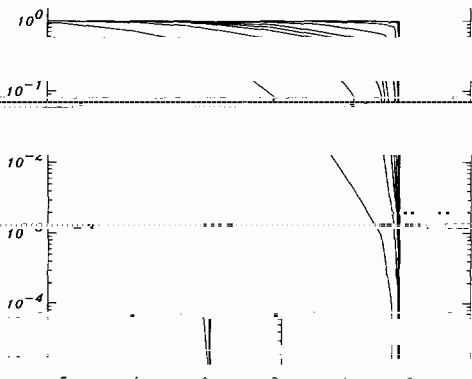


Table 1

Critical values for the semi-standard map.

r_2

$\gamma + \delta$

0.00

ate evaluation of the critical function. For our

r_1 axis appears to be much lower than the value

actually rise rapidly to the correct (actually over-estimated) value as $r_2 \rightarrow 0$. It is interesting that in this case even through the values on axis are

ness implies that the domain of convergence is bounded by the rectangle

$r_1^{(1)} = a^{(2)}(\omega_1)$, independent of r_2 ; the numerical scheme for finding $r_1(r_2)$ when k is small, we

limits to $a^{(2)}(\omega_2)$ on the r_2 axis. This also occurs for the domain of convergence of the second component $B^{(2)}$.

Fig. 8 displays the coefficients $B_n^{(1)}$ and $B_n^{(2)}$ for $s = 10^{-25}$ and $\omega = (\gamma, \sigma)$. In the limit of small slope $B^{(1)} \sim L^{(1)}$ which are the Fourier coeff-

small slope limit of $B_n^{(1)}(s)$. Eq. (38) implies that

upper plot is indistinguishable from that for the

$$B_n^{(1)} \approx B_n^{(2)}$$

spikes and valleys corresponding to a compli-

es, as shown in eq. (38). Using the recursion relation (22) implies that important γ resonance

$$B_n^{(2)}(s) \approx sk \frac{D_{(n,0)}}{D_{(n,1)}} b_{(n,0)}^{(1)} \quad (38)$$

profile approaches a limiting form as $s \rightarrow 0$, even though the magnitude of $B^{(2)}$ approaches zero. Likewise, $B_n^{(2)}$ near the r_2 axis yields the semi- to zero, while similarly converging to a fixed

ing eq. (36) the rules of convergence

$$\frac{1}{D_{(n,0)}} \approx \frac{1}{D_{(n,1)}} \quad (39)$$

As the domain of convergence plots show, the rectangular domain for small k contains the do-

more quickly than for other curves, as we do

for finding $r_1(s)$ has numerical problems when $k \ll 1$. For such small k , the singularity on one axis is dominant over the singularity on the other axis. To illustrate the problem, consider a simple example which has a similar imbalance in the prominence of its singularities. Let

$$\alpha = r_1, \quad B = r_2, \quad \dots$$

Here small values of δ simulate small values of k .

convergence of this series is the rectangle

We examine the behavior of eqs. (34)–(36)

when applied to eq. (59) by a perturbation analysis near $s = 0$. For a finite n , the algorithm gives an error in r_1 of

$$\Delta r_1 \sim -\frac{\delta \alpha}{n} \left(\frac{\alpha s}{B} \right)^n \quad (51)$$

Thus the method works well provided $\delta \ll 1$.

Since the slope is never larger than one, we switch to the inverse of the slope when $s = 1$.

Thus, supposing p is the method fails in a region

below 10^{-5} in the computations

terms of the coupling parameter ϵ , instead of k .

three dimensional graphs seen in figs. 9 and 10.

sum frequency $\omega_1 + \omega_2$ through eq. (47). Numerical overflow for large k prevents us from

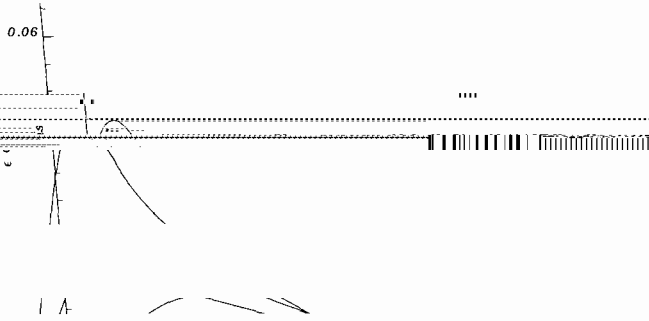


Fig. 9. Radii of curvature curves $r_1(s)$ and $r_2(s)$ for $\epsilon = 10^{-5}$ where r_1 corresponds to the frequency τ_1 and r_2 to τ_2 .

calculating the curves for ϵ too close to its maximum value.

In many ways, it is these three-dimensional

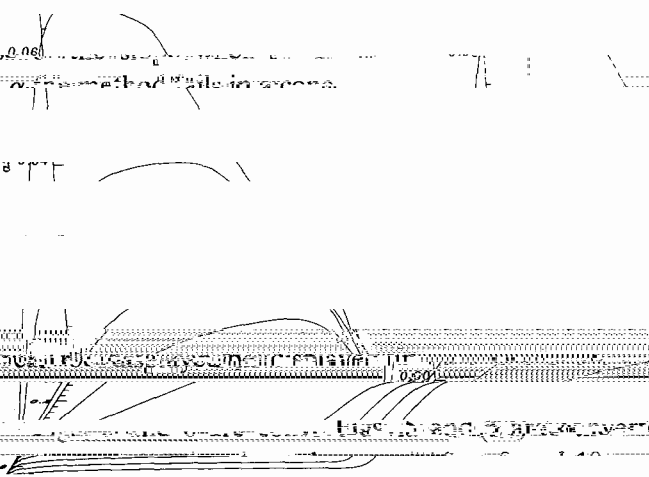


Fig. 10. Same as fig. 9 with $\epsilon = 10^{-4}$.

to determine the "last invariant torus". One

example, the surface for (τ, τ^2) is completely con-

beginning at the origin. One could linearly order

containment of the (γ, ζ) surface is partly due to

definition of order.

to compare the (γ, σ) and (γ, ζ) tori, note that though $a^{ss}(\sigma) > a^{ss}(\zeta)$, $\alpha^{ss}(\gamma + \sigma) < \alpha^{ss}(\gamma + \zeta)$. Thus the surfaces must intersect, and therefore there can only be parametrized comparisons.

Curve based order. An ω torus persists longer than a μ torus along a curve $\xi(t)$ for which $\xi(0) = 0$, if $\xi(t)$ intersects the boundary of the domain of convergence of the μ torus first.

The simplest example of a parameterized family

CONCLUSIONS

example is a particular τ, τ^2 family. For a fixed τ , the surface is contained in the domain of convergence of the μ torus.

We have determined the domain of existence

surface, which is all a curve based order allows. In some sense, one may want to incorporate the

series in the angle variables. The semi-Froeshlé mapping has the advantage that two of the pa-

torus persists longer than a μ torus if the in-

boundary of the domain of several frequency

If one surface is completely contained inside

when projected on the parameters (τ, σ) for

the partial ordering

the uncoupled mappings. Furthermore, numerical results imply that the domain is bounded by

frequencies will intersect in general and then the

eq. (1), they also apply to the $2n$ -dimensional

[10] I.M. Greene and I.C. Percival, Hamiltonian maps in the

[11] J. Guckenheimer, B. Hu and J. Rudnick, Quasiperiodic Transition to Chaos with Three Incommensurate Fre-

tion for four-dimensional volume preserving maps, *Nuovo Cimento* 104B (1980) 177-183

tori of Cubic Irrational Winding Number, University of Houston (1989).

quasiperiodic transitions in the study of dynamical systems. *Phys. Rev. A*

[25] M. Muldoon, Ghosts of Order on the Frontier of Chaos.

circles in area-preserving maps, *Physica D* 7 (1983) 283-300.

[16] R.S. McKay, On Greene's Residue Criterion, *Universi-*

ones, *Kuss. Matn. Surveys* 32:6 (1977) 1-65; G. Benettin, L. Galgani and A. Giorgilli, A proof of Nekhoroshev's theorem for the stability times in nearly

(1992) 149-160.

Physica D 6 (1982) 67-77.