

## Self-consistent chaos in the beam-plasma instability

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The effect of self-consistency on Hamiltonian systems with a large number of degrees of freedom is investigated for the beam-plasma instability using the single-wave model of O'Neil, Winfrey, and Melchior. The single-wave

is observed that the system relaxes into a time asymptotic periodic state where only a few collective degrees are active; namely a clump of trapped particles oscillating in a modulated sea with uniform chaotic sea with oscillations

low degree-of-freedom model is derived that treats the clump as a single *macroparticle*, interacting with the wave and chaotic sea. The uniform chaotic sea is modeled by a fluid waterbag, where the waterbag boundaries correspond approximately to invariant tori. This low degree-of-freedom model is seen to compare well with the simulation.

### 1. Introduction

Chaotic motion in Hamiltonian systems is common whenever there is more than one degree of freedom [1]. Often, the systems studied are low dimensional approximations of many degree-of-freedom systems. In some cases, such

tions can be given with only a few degrees of freedom. However, there are many situations such as galactic dynamics, where the number of degrees of freedom is essentially infinite. Generally, one expects such systems to exhibit greater chaos when the dimension increases;

### 1.1. Self-consistent approximation effects such as “Arnold diffusion”

of degrees of freedom. Even in high dimensional cases it is often of interest to study a low dimensional approximation, for example to study the motion of a single star in a given galactic gravitational potential—this was the motivation for

ferences in [1]). Such an approximation is not self-consistent. Other well studied examples of this type include the motion of charged particles in electromagnetic fields, where the fields produced by the particles are ignored; the motion of tracer particles in a fluid, where the influence of these particles on the fluid velocity field is ignored (the passive advection problem); and

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the so-called “kinematic” dynamo, where a velocity field can intensify a magnetic field, but the back reaction of the field is ignored.

There has been little work on the effect of self-consistency. In this paper we show how it is possible in a system with a large number of degrees of freedom for the inclusion of self-consistency to result in dynamics with “effectively” few degrees of freedom.

Hamiltonian for each particle has one and a half degrees of freedom, and so the motion can be chaotic. However, each of the particles is charged and therefore contributes to the potential—this is the self-consistent effect. In degree of freedom, now, the spatial dependence of the field is given. Thus each particle experi-

extent that the other particles contribute to the single mode of the field. This is in contrast to

particle. This latter case is considerably more difficult.

Models similar to the one described above may be appropriate for many physical situations; for example, a galaxy with a predominantly axisymmetric gravitational potential that is perturbed by a small number of modes, say those corresponding to spiral density waves. Each star contributes to these modes, and also has a possibly chaotic motion in the corresponding field. Similar effects occur for planetary rings, beam-beam interactions in accelerators, tearing

The specific problem we consider is the beam-plasma instability. The formulation is due to O’Neil, Winfrey and Malmberg (hereafter re-

beam of charged particles moves in a background neutral plasma. The system is unstable

to the formation of electrostatic plasma waves. Following [2] we suppose that the most unstable of these waves predominates, and neglect the remainder of the modes—this is easily justified during the linear part of the evolution. OWM showed that the wave grows in amplitude until it traps the beam particles. It then saturates and begins to oscillate in amplitude as the beam particles slosh in the wave potential. At this

We neglect these modes; this is justified, for example, if the system has a finite length, and the sideband wavenumbers are forbidden by periodic boundary conditions.

The oscillations of the single wave have saturated. In their model, the beam was represented by a rigid bar in phase space. When the beam is

tate. Mynick and Kaufman computed the frequency shift and amplitude oscillations of the

of the plasma wave oscillates, the beam particles can experience chaotic motion. They studied the motion of a test particle in a model of this oscillating field and showed that much of the test particle phase space is indeed chaotic. However, there is an island in the phase space where the motion is regular, they noted that some fraction of the beam particles in the numerical experiments of OWM should find themselves in the correct region of phase space to be trapped in this oscillating island. Later Adam, Laval and Mendonça [7] studied a model in which a single

particle, interacts with the plasma wave. As we will show below using the Hamiltonian formulation, this two degree-of-freedom system is integrable

it was shown that the macroparticle system has solutions which correspond to periodic oscilla-

tions of the bunch in the wave.

Related self-consistent problems include the interaction of a single particle with many waves [8] and the interaction of one wave with many other waves [9]. The more complicated

side the oscillating separatrix of the wave. We model these boundaries with sinusoidal curves, an assumption consistent with that of the single mode in the potential. Finally, the frequency shift of the trapped particle oscillations due to

wave-particle turbulence, and it is not clear if the analysis of this paper can give any insight into this case.

In Section 2 we review the derivation of the OWM model, obtaining the Hamiltonian formulation of [5] directly from the Vlasov-Poisson equations. Section 3 discusses numerical solutions of the OWM equations with up to  $10^5$  beam

## 2. Single wave model

O’Neil, Winfrey, and Malmberg (OWM) [2] introduced the single wave model to describe the growth and saturation of the weak beam-plasma instability. In this section we briefly review the

and oscillates about a finite amplitude. We observed at least 100 periods of these oscillations; as far as we can determine, the oscillations persist and the system becomes asymptotically

and obtain the Hamiltonian form previously presented by Mynick and Kaufman [5], discuss linear instability, and finally consider a special case where only a single beam particle is included

ticle in this periodic potential, showing that a substantial portion of the original beam is indeed trapped in a stable island in the test particle

### 2.1. Derivation

To obtain the single wave model, the response of the beam and plasma to the field is treated separately. We consider only the one dimensional, collisionless, nonrelativistic, electrostatic case. The total electron density

beam finds itself in the chaotic region of phase space, and spreads more or less uniformly over this region. The upper and lower boundaries of this “chaotic sea” are formed from invariant tori of the test particle system.

$$n(x, t) = n_p(x, t) + n_b(x, t) = \sum_{j=1}^N f_j(x, t) \quad (1)$$

In Section 4, we construct a four degree of freedom model. One degree of freedom is the wave, the second corresponds to the trapped

is a sum of contributions from the plasma and

tions of the boundary of the chaotic sea and are derived from the “waterbag” approximation.

linear force equation

$$m\ddot{x}_j = -eE(x_j, t), \quad (2)$$

A waterbag consists of a constant phase space

where the electron charge is

case the simulations show that the phase space density of the chaotic particles is indeed nearly constant and the boundaries of the chaotic zone are formed from invariant surfaces well out-

phase velocity of the resulting instability is much larger than the velocities of particles in the background plasma: the plasma responds nonresonantly, and trapping effects of plasma particles

in the wave can be neglected. This implies that  $\hat{\epsilon} \approx 1$ . At this point we assume that the electrostatic

$$4\pi en_p(x, t) = (1 - \hat{\epsilon})\varphi''(x, t), \quad (3)$$

where  $\varphi$  is the electrostatic potential,  $E = -\varphi'$ . Substituting this into Poisson's equation for  $\hat{\epsilon}$

$$\hat{\epsilon}\varphi = 4\pi en_b(x, t). \quad (4)$$

tion is most easily treated by Fourier transform. In this representation the dielectric parameter becomes a real function,  $\epsilon(k, \omega)$ . For a weak beam,

that the electrostatic response is dominated by

is a reasonable approximation to expand  $\epsilon$  about one such zero retaining only the first derivative of  $\epsilon$  with respect to  $\omega$ :

$$\epsilon(k, \omega) \approx \epsilon(k, \omega_0) + \left. \frac{\partial \epsilon}{\partial \omega} \right|_{\omega=\omega_0} (\omega - \omega_0) = \epsilon'(\omega - \omega_0). \quad (5)$$

For example, for a cold plasma  $\epsilon = 1 - \omega_p^2/\omega^2$ , and  $\partial\epsilon/\partial\omega|_{\omega=\omega_0} \equiv \epsilon' = 2/\omega_p$ . Transforming back to the time domain and using Eq. (4) then gives

$$\dot{E}_k + i\omega_0 E_k = \frac{4\pi e}{kL\epsilon'} \sum_{j=1}^N e^{-ikx_j(t)}, \quad (6)$$

where we have used the Fourier transform of the beam density of Eq. (1):

$$n_b(x, t) = \frac{1}{L} \sum_{j=1}^N e^{-ikx_j(t)}, \quad (7)$$

$k$ -space is relatively narrow in units of  $2\pi/L$ . In this case, if  $k$  represents the most unstable mode, the amplitude of all other Fourier com-

single wave during the linear growth stage. Of course, some time after nonlinear saturation of

stable spectrum depends on the small parameter  $(n_b/n_p)^{1/3}$ , so that the single wave model will be most appropriate in the weak beam case.

Under the single wave assumption, Eq. (2) yields

$$\dot{p}_j = -e(E_k e^{ikx_j} + E_{-k} e^{-ikx_j}). \quad (8)$$

Equation (8) together with Eq. (6) are the closed dynamical system that governs the interaction of a single wave with the beam particles.

## 2.2. Hamiltonian structure and derivation

Now consider the derivation of the equations of motion, Eqs. (6) and (8), within the Hamiltonian context. The derivation proceeds by first considering the kinematics, i.e. the dynamical variables used to describe the state of the system, and then the dynamics, obtained by finding the appropriate Hamiltonian.

We begin the first part by supposing that the electrons are described by specifying their

$1, 2, \dots, M$ . The first  $N (< M)$  of these particles are singled out to represent the beam dynamics, while the remaining  $M - N$  particles represent the background plasma. The phase space density, i.e. distribution function, can be

$$f(x, p, t) = f_b(x, p, t) + f_p(x, p, t) + \sum_{j=N+1}^M \delta(x - x_j(t)) \delta(p - p_j(t)) \quad (9)$$

$$= \int \frac{\delta G}{\delta f_p} \sum_{j=N+1}^M \left( \frac{\delta f_p}{\delta x_j} \delta x_j + \frac{\delta f_p}{\delta p_j} \delta p_j \right) dx dp$$

The Poisson bracket in terms of  $(x_j, p_j)$  where  $j = 1, 2, \dots, M$ , has the standard canonical form

$$[g, h] = \sum_{j=1}^M \left( \frac{\partial g}{\partial x_j} \frac{\partial h}{\partial p_j} - \frac{\partial h}{\partial x_j} \frac{\partial g}{\partial p_j} \right) \equiv [g, h]_b + [g, h]_p, \quad (10)$$

where  $g$  and  $h$  are functions defined on phase space.

It is desired to describe the state of system in terms of the above phase space coordinates for the beam particles. However, for the background plasma, the phase space coordinates of these particles will be replaced by a Vlasov type distribution function,  $f_p$ . This can be achieved by mapping the Poisson bracket of Eq. (10) to these variables; but  $f_p$ , unlike  $(x_j, p_j)$ , is not a canonically conjugate set of coordinates, i.e.  $f_p$  is a *noncanonical* variable, therefore the resulting Poisson bracket possesses noncanonical form [11]. In order to effect this transformation the chain rule [12,13] for functional derivatives is required. Suppose

$$g(x_j, p_j) = G[f_p], \quad (11)$$

where  $j = N + 1, N + 2, \dots, M$ . Here  $G[f_p]$  is a *functional* of  $f_p$ ; the relationship between the phase space function  $g$  and the functional  $G$  is

obtained by varying both sides of this equation:

$$\delta g = \sum_{j=N+1}^M \left( \frac{\partial g}{\partial x_j} \delta x_j + \frac{\partial g}{\partial p_j} \delta p_j \right) = \delta G$$

The operators  $\delta f / \delta x_j$  and  $\delta f / \delta p_j$  are obtained by variation of Eq. (9) with respect to the plasma particles and denote the formal adjoint. The result is finally

$$\frac{\partial g}{\partial x_j} = \int \left( \frac{\partial}{\partial x} \frac{\delta G}{\delta f_p} \right) \delta(x - x_j) \delta(p - p_j) dx dp = \left( \frac{\partial}{\partial x} \frac{\delta G}{\delta f_p} \right) |_{(x_j, p_j)}$$

$$\frac{\partial g}{\partial p_j} = \int \left( \frac{\partial}{\partial p} \frac{\delta G}{\delta f_p} \right) \delta(x - x_j) \delta(p - p_j) dx dp = \left( \frac{\partial}{\partial p} \frac{\delta G}{\delta f_p} \right) |_{(x_j, p_j)} \quad (13)$$

Insertion of Eq. (13) into the second term of Eq. (10) yields the bracket

$$[G, H] = \int f_p \left\{ \frac{\delta G}{\delta f_p}, \frac{\delta H}{\delta f_p} \right\} dx dp + \sum_{j=1}^N \left( \frac{\partial G}{\partial x_j} \frac{\partial H}{\partial p_j} - \frac{\partial H}{\partial x_j} \frac{\partial G}{\partial p_j} \right), \quad (14)$$

where

$$\{g, h\} \equiv \frac{\partial g}{\partial x} \frac{\partial h}{\partial p} - \frac{\partial h}{\partial x} \frac{\partial g}{\partial p}. \quad (15)$$

Here the quantities  $G$  and  $H$  are functionals of  $f_p$ , but according to Eq. (9) they can be thought of as ordinary functions of the beam particle coordinates  $(x_j, p_j)$  where  $j = 1, 2, \dots, N$ . Note that discreteness has now disappeared from  $f_p$ .

ground Vlasov plasma electrons is obtained by inserting

$$f(x, p, t) = f_p(x, p, t) + \sum_{j=1}^N \delta(x - x_j(t)) \delta(p - p_j(t)) \quad (16)$$

into the following energy expression:

$$H[f_p; x_j, p_j] = \frac{1}{2m} \int p^2 f_p dx dp - \frac{e}{2} \int f_p \phi_p dx + \sum_{j=1}^N \left( \frac{p_j^2}{2m} - e\phi_p(x_j) - \frac{1}{2}e\phi_b(x_j) \right), \quad (18)$$

where  $\phi_p(x_j)$  and  $\phi_b(x_j)$  are the contributions to the electrostatic potential of the plasma and

yields the hybrid system.

Now we can turn to the task of obtaining, from the hybrid system, the approximate system of

summed to be described by an equilibrium distribution function of compact support in velocity, plus the single linear wave, whose phase velocity

wave-particle effects are eliminated in the back-

tion of the distribution function, the analysis of [14] and [15] implies that the linearization of the plasma energy becomes identically the well-known expression for the dielectric energy of a plasma wave. Second, the self-interaction potential of the beam,  $\phi_b$ , is neglected in comparison to that of the plasma,  $\phi_p$ , a justifiable assumption in light of smallness of  $n_b/n_p$ . Thus, Eq. (18) becomes

$$H(E_k, E_{-k}, x_j, p_j) = \frac{L}{4\pi} \omega_0 \epsilon' |E_k|^2 + \sum_{j=1}^N \left( \frac{p_j^2}{2m} - \frac{ie}{k} E_k e^{ikx_j} + \frac{ie}{k} E_{-k} e^{-ikx_j} \right). \quad (19)$$

It remains to find the appropriate Poisson bracket in terms of  $E_k$  and  $E_{-k}$  instead of  $f_p$ . Since the plasma is in essence being modeled as a fluid, an easy way to obtain this is to map

the first term of Eq. (14) to fluid variables as

$$[g, h] = \sum_{j=1}^N \left( \frac{\partial g}{\partial x_j} \frac{\partial h}{\partial p_j} - \frac{\partial h}{\partial x_j} \frac{\partial g}{\partial p_j} \right) - \frac{i4\pi}{L\epsilon'} \left( \frac{\partial g}{\partial E_k} \frac{\partial h}{\partial E_{-k}} - \frac{\partial h}{\partial E_k} \frac{\partial g}{\partial E_{-k}} \right). \quad (20)$$

Eqs. (6) and (8) in the form

The bracket of Eq. (20) is not quite canonical; however, with the substitution

$$E_{-k} = i \left( \frac{4\pi}{L\epsilon'} \right)^{1/2} \mathcal{J}^{1/2} e^{i\vartheta}, \quad (22)$$

the electric field is expressed in terms of conventional action-angle variables, and the bracket as

$$[g, h] = \sum_{j=1}^N \left( \frac{\partial g}{\partial x_j} \frac{\partial h}{\partial p_j} - \frac{\partial h}{\partial x_j} \frac{\partial g}{\partial p_j} \right) + \left( \frac{\partial g}{\partial \vartheta} \frac{\partial h}{\partial \mathcal{J}} - \frac{\partial h}{\partial \vartheta} \frac{\partial g}{\partial \mathcal{J}} \right), \quad (23)$$

while the Hamiltonian of Eq. (19) becomes

$$H(\vartheta, \mathcal{J}, x_j, p_j) = \omega_0 \mathcal{J} + \sum_{j=1}^N \left[ \frac{p_j^2}{2m} - \frac{2e}{k} \left( \frac{4\pi}{L\epsilon'} \right)^{1/2} \mathcal{J}^{1/2} \cos(kx_j - \vartheta) \right]. \quad (24)$$

To complete the derivation, it is convenient to introduce scaled, dimensionless variables based on the fundamental frequency,

$$\omega_b^3 = \frac{4\pi e^2 N}{mL\epsilon'}. \quad (25)$$

Here  $\omega_b$  is a harmonic mean of the beam's plasma frequency and  $1/\epsilon'$ , which is of order

the background plasma frequency. Note that the small parameter  $(n_b/n_p)^{1/3}$  is represented by the ratio  $\omega_b/\omega_p$ . By a sequence of time depen-

$$\frac{d\Phi}{d\tau} = [\Phi, H] = [\Phi, \Phi^*] \frac{\partial \Phi}{\partial \Phi^*}, \quad (31)$$

which together with the canonical bracket for

$$H(J, \theta, p_j, \xi_j) = \sum_{j=1}^N \left[ \frac{1}{2} p_j^2 - 2 \left( \frac{J}{N} \right)^{1/2} \cos(\xi_j - \theta) \right], \quad (26)$$

where the dimensionless variables are defined by

$$\begin{aligned} \frac{d\Phi}{d\tau} &= \frac{i}{N} \sum_{j=1}^N e^{-i\xi_j}, \\ \frac{d^2\xi_j}{d\tau^2} &= i\Phi e^{i\xi_j} - i\Phi^* e^{-i\xi_j}. \end{aligned} \quad (32)$$

Note that these equations hold for arbitrary choices of the physical parameters, such as  $e/m$  and the beam density; it is only the relationship between scaled variables and physical variables which changes. (Of course, the single wave

$$k^2 \dots \dots \dots (27)$$

is canonically conjugate to  $p_j$ , is defined in a frame moving at the phase velocity  $\omega_0/k$ . This Hamiltonian is (in general) non-integrable.

symmetry (we don't know of any nontrivial symmetries) which is the translation,  $\xi_j \rightarrow \xi_j + \epsilon$ ,  $\theta \rightarrow \theta + \epsilon$ . This implies that the total mo-

It is often convenient to use a noncanonical wave amplitude variable instead of action-angle variables. This is easily done if we use as inde-

$$P \equiv \sum_{j=1}^N p_j + J \quad (33)$$

pendent variables the action-angle variables and the complex conjugate  $\Phi^*$  defined by

is concerned. To take advantage of this, define the canonical transformation generated by  $F_2 = P\theta + \sum p'_j (\xi_j - \theta)$  which gives the new momenta, ( $p'_j \equiv p_j, P$ ), and angles, ( $\psi_j \equiv \xi_j - \theta, \theta$ ). The new Hamiltonian is

$$\Phi(\tau) = \left( \frac{J}{N} \right)^{1/2} e^{-i\theta}. \quad (28)$$

In these coordinates the Poisson bracket be-

$$= \sum_{j=1}^N \left[ \frac{1}{2} p_j^2 - \frac{1}{\sqrt{N}} \left( P - \sum_{k=1}^N p_k \right) \cos \psi_j \right], \quad (34)$$

$$[\Phi^*, \Phi] = 1/N \quad (29)$$

and the Hamiltonian takes the form

$$H = \sum_{j=1}^N \left( \frac{1}{2} p_j^2 - \Phi e^{i\xi_j} - \Phi^* e^{-i\xi_j} \right). \quad (30)$$

which has effectively  $N$  degrees of freedom since  $\theta$  is a constant of motion.

the first term is the particle kinetic energy and the last two represent the electrostatic potential energy. The equations of motion are obtained from the Poisson bracket,

### 2.3. Linear instability

To establish the fact that the Hamiltonian of Eq. (30) properly describes at least the linear

stage of the weak beam-plasma instability, con

As will be seen in the next section, the linear

0. Linearizing the Hamiltonian about this equi-

$$H = \sum_{j=1}^N \left[ \frac{1}{2} p_j^2 - 10\Phi \delta \xi_j e^{-i\xi_j} + 10\Phi^* \delta \xi_j e^{i\xi_j} \right]. \tag{35}$$

The resulting linear equations of motion can be straightforwardly diagonalized to obtain the characteristic polynomial  $\omega^{2N-4} (\omega^6 - 1) = 0$ , where the solutions have been assumed to possess

required because of the Hamiltonian nature of the flow (recall that if  $\omega$  is an eigenvalue then  $\omega^*$ ,  $-\omega$  and  $-\omega^*$  must also be eigenvalues). In dimensional units, using Eq. (27), we have

$$\hat{\omega}_j = \omega_b e^{ij\pi/3}, \quad j = 0, 1, \dots, 5, \tag{36}$$

which includes the unstable beam-plasma mode (the case  $j = 2$ ). We can physically identify the eigenmodes by considering the equations of motion. Differentiating the equation for  $\Phi$  twice and substituting for  $\xi$  gives

$$\frac{d^2 \Phi}{d\tau^2} = \frac{N}{N} \sum_{j=1}^N e^{-ik_j} \delta \xi_j = 10\Phi, \tag{37}$$

upon noting that  $\sum e^{-2ik_j} = 0$ . This shows that the nonzero frequencies are associated with nonzero  $\Phi$ . The eigenmodes for the conjugate roots,  $\omega^*$ ,  $-\omega$  and  $-\omega^*$ , are the same as that for  $\omega$  except for varying choices of signs. The remaining roots of the characteristic equation ( $\omega = 0$  of multiplicity  $2N - 4$ ) have eigenmodes with  $\delta\Phi = \delta\Phi^* = 0$  and positions given by the  $N - 2$  independent solutions of  $\sum e^{-ik_j} \delta \xi_j = 0$ . The double multiplicity of each of these roots arises from allowing the momenta to be nonzero.

The only exactly-solvable case of the Hamiltonian Eq. (26) appears to be that of a single par-

tum, Eq. (33), reduces this case to one effective degree of freedom, and it can be integrated by quadrature. When  $N$  is larger than unity we expect to lose integrability. It will be of interest later to consider the case of  $N_m$  particles clumped

$$H = \frac{p^2}{2N_m} - 2N_m \left( \frac{J}{N} \right)^{1/2} \cos(\xi - \theta), \tag{38}$$

where  $p = \sum_{i=1}^{N_m} p_i = N_m p_2 \dots$  is the macroparticle momentum. The Hamiltonian  $H$  can be reduced to one degree of freedom by defining the total momentum  $P = p + J$  as before to obtain

$$H = \frac{p^2}{2N_m} - 2N_m \left( \frac{P-p}{N} \right)^{1/2} \cos \psi. \tag{39}$$

The equations for this case were studied in detail by Adam, Laval and Mendonca [7], who did not use the Hamiltonian approach.

The Hamiltonian Eq. (39) has isolated, nondegenerate fixed points. These occur at the points defined by

$$p_0^3 - p_0^2 P + N_m^3 \frac{N_m}{N} = 0, \quad \psi_0 = 0 \text{ or } \pi. \tag{40}$$

The fixed point with  $(p_0 < 0, \psi_0 = 0)$  is stable and corresponds to the macroparticle sitting in the bottom of the potential well. The two fixed points with  $(p_0 > 0, \psi = \pi)$  are less intuitive. These exist only if  $P > 3N_m (N_m/4N)^{1/3}$ . They correspond to a particle perched on the top of the potential well. The lower momentum particle is unstable, while the larger momentum particle is stable.



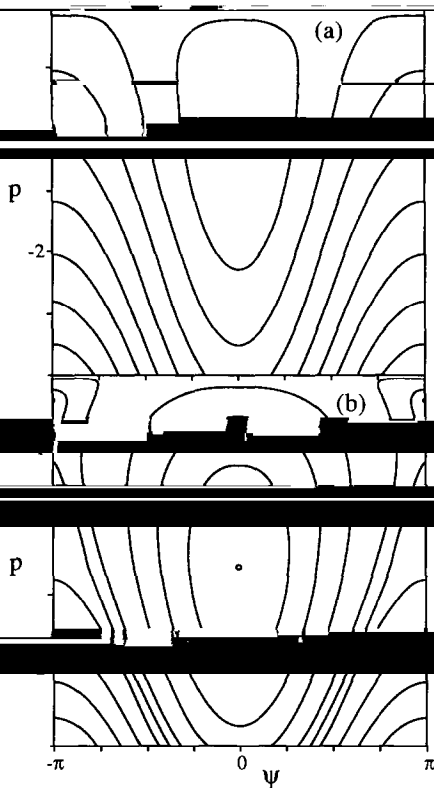


Fig. 1. Contours of  $H$ , Eq. (39), in the single particle phase space  $(p, \psi)$ . Units of  $p$  (and  $P$ ) are  $N_m(N_m/N)^{1/3}$ . In the upper figure  $P = 0$  and there is one fixed point; in the lower figure  $P = 2$  and there are three fixed points.

space. Small oscillations about the stable fixed point are such that as the particle starts to fall to catch the particle.

The final "fixed point" is the degenerate case ( $p = P, \psi = \text{arbitrary}$ ) for which the wave amplitude is zero. This corresponds to the unstable

bility results, as discussed in the previous section.

$$M_e \equiv N_m \frac{2J_0}{2J_0 - p_0} \tag{41}$$

Here  $M_e$  is the effective mass due to the self-consistent coupling. Note that since  $n_0 < 0$ , the

in a fixed potential well. This gives a bounce frequency of

$$\omega_B^2 = 2 \left( \frac{J_0}{N} \right)^{1/2} \left( 1 - \frac{p_0}{2J_0} \right) \tag{42}$$

The first factor is just the bounce frequency of a particle in a fixed well, normalized in accord with Eq. (27). The latter term provides an increase in the frequency, as also discussed in [5].

To compare these with the simulations in the next section, we set the total momentum,  $J$ , to zero, corresponding to the initial conditions  $J(0) = p(0) = 0$ . In this case  $J(\tau) = -p(\tau)$  and there is only one fixed point

$$\Omega \equiv \frac{p_0}{N_m} = - \left( \frac{N_m}{N} \right)^{1/2}, \quad \psi = 0. \tag{43}$$

From the simulations we will find  $N_m \simeq 0.4N$ , and thus that

$$\Omega = -0.74, \quad |\Phi| = 0.29, \quad \omega_B = 0.94. \tag{44}$$

### 3. Simulations

Simulations of the model of Eqs. (32) were carried out for several initial conditions and the results shown below were obtained with a symplectic, leap frog method.

#### 3.1. OMM model

In the simulations the particles are initialized as a cold beam with uniform spacing  $\xi$

with  $N = 10^9$ . For small  $\tau$  the wave ampli-

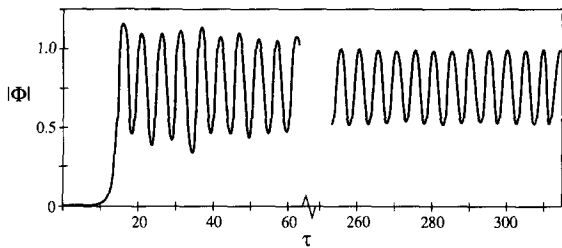


Fig. 2. Plot of  $|\Phi(\tau)|$ , the normalized wave amplitude, for  $N = 10000$  particles initialized as a cold beam.

predicted by Eq. (36) with the phase velocity  $v_\phi = \mathcal{R}(e^{2\pi i/3}) = -0.5$ . As the wave grows, the beam experiences a growing sinusoidal perturbation, and as can be seen in the density plot of Fig. 3, the beam density also varies sinusoidally.

begin to oscillate in the wave. Consequently the amplitude of the wave reaches a maximum.

At this stage the beam density is  $(N_e) = 1$ .

and undoubtedly greatly change the subsequent behavior of the system.

None-the-less, the subsequent development of the OWM dynamics is quite interesting. As the beam particles begin to oscillate in the wave, their oscillation frequencies depend upon their energy, just as for a single particle in a fixed potential. Thus as the beam begins to rotate about the potential minimum, those particles closer to the center have larger oscillation frequencies than those near the "separatrix".

If the wave amplitude were fixed, one would see phase mixing of the particles (visualized as an ever tighter spiral in the particle phase space), and the oscillations in the particle total energy would damp away—this is the mechanism of Landau damping in a large amplitude

However since  $v_\phi \approx -0.5$  and the beam is initialized at  $v = 0$ , when the beam particles oscillate in the wave, their net momentum also oscillates.

the single-wave model does not allow the devel-

amplitude is not fixed and each beam particle

These would lead to the growth of other waves

known, the phase space for a single beam par-

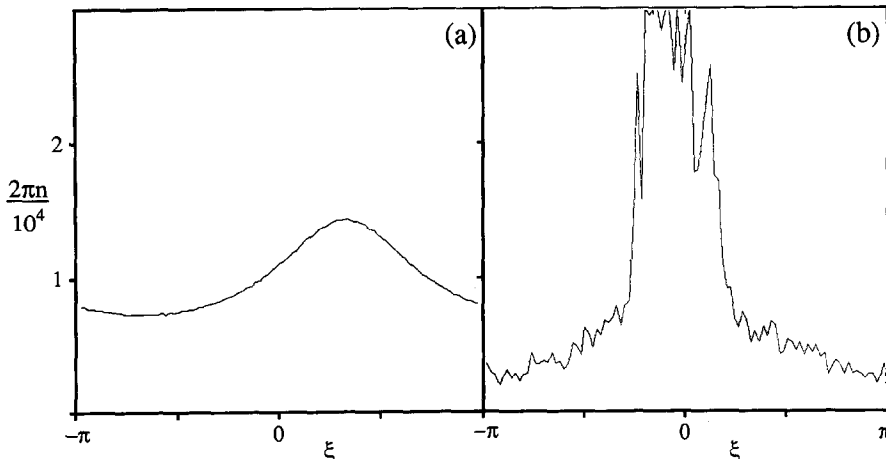


Fig. 3. Plot of the beam density as a function of position. The sinusoidal distortion of the density due to the growing wave is observed in (a) at  $\tau = 12.6$ . By  $\tau = 60.2$  in (b), about 10 bunches have occurred and the macro-particle has formed (represented here by the sharp peak around  $\xi = 0$ ). The remaining chaotic sea particles have a sinusoidal density variation

Fig. 4. Plot of the beam particle phase space at  $\tau = 641$  showing a well defined macroparticle and chaotic sea.

of the beam particles at a fixed time. Note the

with a nearly uniform density, and the other a more coherent cluster of particles. In the cluster one still sees evidence of the initial beam, though

In the simulations which were carried out up-  
 sisted, and indeed, as can be seen in Fig. 2 the oscillations become increasingly periodic with time. Furthermore as we varied  $N$  up to  $10^5$  and improved the integration accuracy, we noticed that these oscillations became more periodic and constant in amplitude as the number of particles increased and as the accuracy improved. Thus we believe that the asymptotic state is a periodic one.

Meanwhile, the particle phase space exhibits quite complicated behavior. About 60% of the particles—those with relatively large energies in the wave frame—experience chaotic motion, and spread out roughly uniformly in a region of phase space whose average width is  $\Delta\omega = 4.7$ . We called the chaotic particles the “stochastic

“chaotic sea” The remaining 40% of the parti-

wave as seen in the sequence of phase space instantaneous snapshots of the wave, that is

of the cluster oscillate  $180^\circ$  out of phase. This clump of particles is treated as a single particle, the macroparticle, in the model of SECTION 4.

In addition to the uniform cold beam, several different initial conditions have been partially studied. For example, a nonuniform cold beam, a circular ring,  $\xi + p = r$ , and also considered these cases some subset of the particles remained

system did not appear to settle into an equilibrium. Cold beams with nonzero momenta also lead to oscillations as was shown in [3], though

the momentum. We have not investigated this in initial conditions that will give rise to a periodic final state.

### 3.2. Test particle

To investigate further dynamics of the beam particles, consider the “test particle” motion of a single particle in a given oscillating potential. This is obtained from the nonself-consistent, one and one half degree-of-freedom Hamiltonian

$$H_t(p, \xi, \tau) = \frac{1}{2}p^2 - 2 \left( \frac{J(\tau)}{N} \right)^{1/2} \cos(\xi - \theta(\tau)), \tag{45}$$

where  $L$  and  $\theta$  are considered to be given vari-

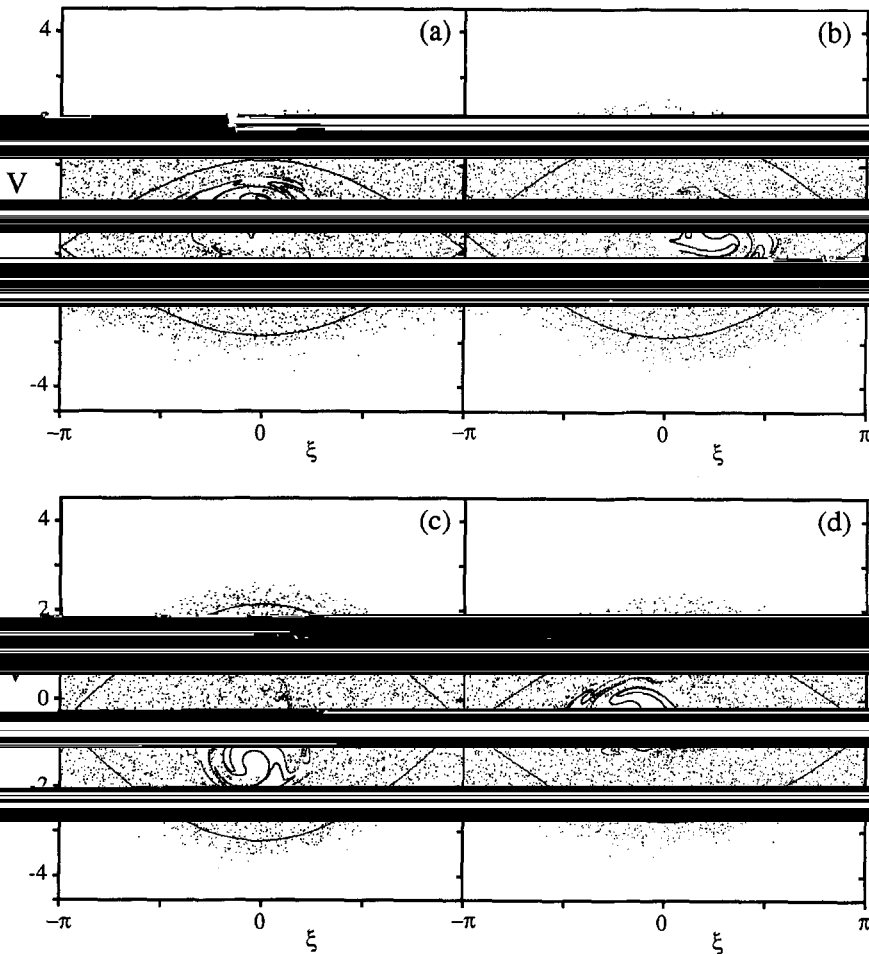


Fig. 5. Sequence of beam phase space plots over one bounce period. Note that the macroparticle bounces coherently in the wave, and the wave amplitude and chaotic sea boundaries also oscillate periodically.

Here we determine  $J$  and  $\theta$  numerically, from the simulations of Section 3.1, building these functions from an average over a number of pe-

A stroboscopic plot of the test particle dynamics is shown in Fig. 6 for several different values of  $\theta$ . The dots represent the trajectories of a number of different test particles. As was also noted in [6], there is a prominent stable island in the test particle phase space which oscillates exactly out of phase with the potential; much of the rest of the phase space is chaotic. Also shown

in the plots represents the position of one of the 10 000 beam particles. Note that the macroparticle clump sits, as near as can be ascertained

verifies an assertion in [6], where it was merely noted that some fraction of the beam particles initially stretched across the position of the test particle island.

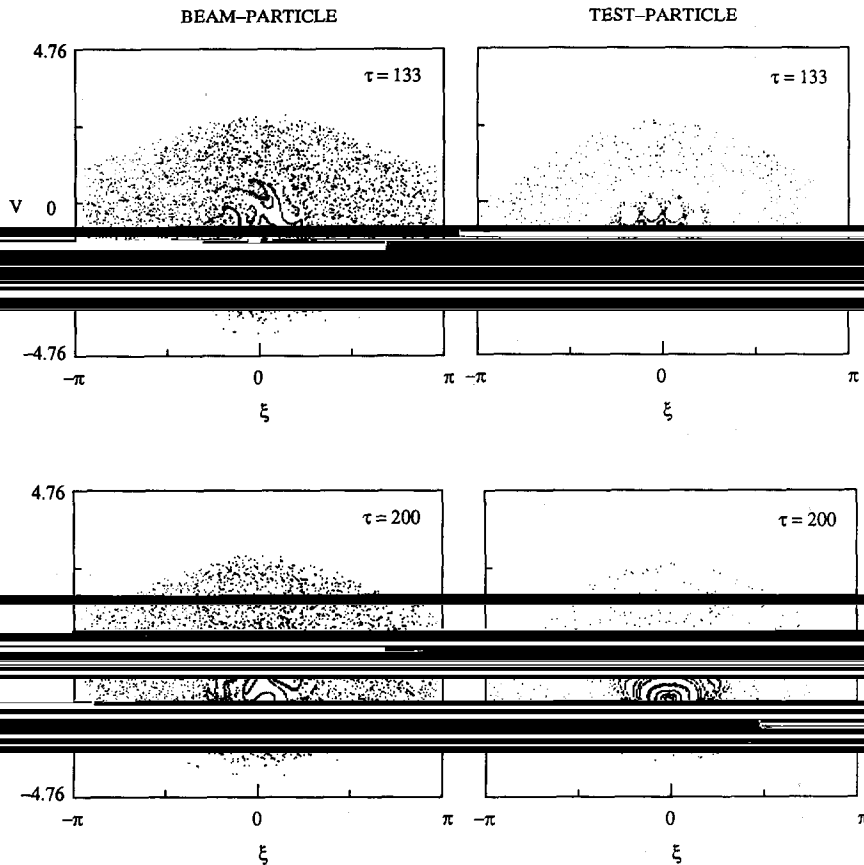


Fig. 6. Phase space plots comparing the full simulation of Section 3 with the dynamics of a test particle in a given time-dependent potential  $\Phi(\tau)$  as determined by the simulation. Shown are several test particle initial conditions at three different times during a cycle.

#### 4. Chaotic sea model

As we have seen from the simulations, the asymptotic state of the cold beam initial condition, evolved under the OWM Hamiltonian, appears to be almost exactly periodic. Approximately 10% of the initial beam forms a clump of particles that oscillates in the potential well formed by the wave. The remaining particles

freedom, which approximately describes the full 10 001 degree-of-freedom system. In the model, as noted above, we assume that the clump of regularly oscillating particles is localized enough so that all these particles can be treated as one located at  $(\xi, p)$ . This macroparticle contains  $N_m$  particles and hence a mass  $mN_m$  and charge  $-eN_m$ . The approximation that the  $N_m$  particles can be treated as a single particle ignores any in-

In this section we use these ideas to obtain a reduced Hamiltonian model of four degrees of

be similar to the analysis of Myhrer and Kadman [5] who assumed that the cluster of parti-

cles formed a "bar" in phase space (an assumption that is reasonable only for the case of a

where  $v_{\pm}^0$  are the mean velocities (which do not depend on time), and  $\tilde{v}_{\pm}$  are the fluctuations.

tion Eq. (45) about the central periodic orbit. The condition of an oscillator degree of freedom to the

the boundary condition is small. This is the condition of a chaotic sea. The evolution of the field is obtained from

Much more interesting is the treatment of the phase space density of these particles appears to

density of the chaotic sea, as given by Eq. (46)

in velocity space. We assume that these particles can be treated as a continuum with a constant phase space density  $f_c$  between the boundaries

$$\frac{m\omega}{k\epsilon'} (f_c(\tilde{v}_+ - \tilde{v}_-) + \frac{N_m}{L} e^{-ikx_m}). \quad (50)$$

These equations are nondimensionalized as

$$n_c(x, t) = \int_{v_-}^{v_+} f_c dv = f_c(v_+ - v_-) \quad (46)$$

$$\omega_{\pm} \equiv \frac{kv_{\pm}^0 - \omega_0}{\omega_b}, \quad V_{\pm} \equiv \frac{k}{\omega_b} \tilde{v}_{\pm} e^{i\omega_0 t}. \quad (51)$$

is the density of the chaotic sea. The total number of such particles in the length  $L$  will be denoted by  $N_c = N - N_m$ . Particles in the chaotic sea evolve according to Eq. (2), and hence  $f_c$  evolves according to the Vlasov equation. As is well known, and easy to see, the Vlasov equation

In terms of these variables the equations of motion become

$$\begin{aligned} \dot{\Phi} &= i \frac{N_c}{N\Delta\omega} (V_+ - V_-) + i \frac{N_m}{N} e^{-i\zeta}, \\ \dot{\zeta} &= i\Phi e^{i\zeta} - i\Phi^* e^{-i\zeta}, \end{aligned}$$

two equations for the evolution of the boundaries [18]. These equations are called the waterbag equations:

where  $\Delta\omega = \omega_+ - \omega_-$  is the average, nondimensional width of the chaotic sea.

$$\begin{aligned} \frac{\partial v_+}{\partial t} + v_+ \frac{\partial v_+}{\partial x} &= -eE, \\ \frac{\partial v_-}{\partial t} + v_- \frac{\partial v_-}{\partial x} &= -eE. \end{aligned} \quad (47)$$

This set of equations is also a Hamiltonian system, with the wave action-amplitude variables defined in Eq. (28), and the new action-amplitude variables for the chaotic sea defined by

Following the philosophy of the derivation of the QNM model, where only a single wave number

$$V_{\pm} = \left( \frac{J_{\pm} \Delta\omega}{N} \right)^{1/2} e^{-i\theta_{\pm}}, \quad (52)$$

in  $v_{\pm}(x)$ :

$$v_{\pm} = v_{\pm}^0 + \tilde{v}_{\pm} e^{ikx} + \tilde{v}_{\pm}^* e^{-ikx}. \quad (48)$$

The equations of motion then become

which results in the Poisson bracket relations

$$\left( \frac{\partial}{\partial t} + ikv_{\pm}^0 \right) \tilde{v}_{\pm} = -eE_k, \quad (49)$$

$$[V_{\pm}^*, V_{\pm}] = \pm i \frac{\Delta\omega}{N_c}. \quad (54)$$

The Hamiltonian takes the form

$$H = \frac{p^2}{2N} + \frac{N_c \omega_+}{\Delta\omega} |V_+|^2 - \frac{N_c \omega_-}{\Delta\omega} |V_-|^2 + N_m \Phi^* e^{-i\xi} + \text{c.c.}$$

In terms of canonical variables this becomes

$$H = \frac{p^2}{2N_m} + \omega_+ J_+ - \omega_- J_- - \sqrt{J_+ J_-} \cos(\theta_+ - \theta_-)$$

where the coefficients are given by

$$\alpha = \left( \frac{N_c}{N\Delta\omega} \right)^{1/2}, \quad \beta = \frac{N_m}{\sqrt{N}}. \tag{57}$$

The first three terms in the Hamiltonian repre-

of macroparticle energy and harmonic terms for the oscillations of the chaotic sea boundary. The last three terms give the electrostatic interaction energy.

Thus we have reduced the 10001 degree-of-freedom Hamiltonian to one of four degrees of freedom, which describes well the motion in the periodic final state of the simulations, *provided* the three parameters  $\omega_+$ ,  $\omega_-$  and  $N_m$  are given.

total momentum given by,

$$P = J_+ - J_- + N(\omega_+ - \omega_-) \tag{58}$$

tum in the wave and macroparticle, and the last three are the contributions of the chaotic sea. These latter terms include the momentum in the oscillations of the waterbag boundary,  $J_+ - J_-$ ,

tion can also be derived from Eq. (33) by splitting off the contribution of the particle momen-

served quantities besides the energy. Thus the and should exhibit the full complexity of such diffusion.

The stable equilibrium of Eq. (56) corre-

$$J_{\pm} = \frac{\alpha^2}{(\Omega - \omega_{\pm})^2} J. \tag{59}$$

From the simulation results of Fig. 5, we found  $N_m \approx 0.4N$ ,  $\omega_+ \approx 1.9$ , and  $\omega_- \approx -2.8$ . This gives  $\alpha = 0.36$ ,  $\beta = 40$ , and  $\Delta\omega = 4.7$ . For these parameters we can solve Eq. (59) to obtain

$$\Omega = -0.66, \quad |\Phi| = 0.73, \\ |V_+| = 0.29, \quad |V_-| = 0.35. \tag{60}$$

On the other hand, using Fig. 5 to determine the average phase velocity of the wave in the simulations we obtain  $\Omega \approx -0.67$ . The average value of  $|\Phi|$  from Fig. 2 is 0.75. Both of these

Eq. (11). In fact, if we take the measured value of  $\Omega$  as given, then Eq. (59) gives the macroparticle

To compute the frequency of small oscillations about the equilibrium, we first eliminate the action  $J$  using the conservation law (58), defining phases  $\psi = \theta - \xi$ , and  $\psi_{\pm} = \theta_{\pm} \mp \theta$ , conjugate to the momenta  $p$  and  $J$ , respectively. Using the

$$\delta^2 H = \frac{1}{2} \delta n \cdot M^{-1} \cdot \delta n + \frac{1}{2} \delta \psi \cdot K \cdot \delta \psi \quad (61)$$

$\delta \psi_+, \delta \psi_-$ ) are the deviations from equilibrium. The matrix  $M$ , the effective mass matrix, is positive definite. The matrix  $K$ , the effective spring constant matrix, turns out to be diagonal. In order that the three terms in  $K$  be positive, we must assume that  $\psi = \psi_+ = 0$ , while  $\psi_- = \pi$ . This is consistent with the fact that the lower boundary of the chaotic sea is observed to have a  $180^\circ$  phase lag with respect to the upper boundary.

The frequencies of small oscillation are given by the square roots of eigenvalues of the matrix  $KM^{-1}$ . For the parameters of the simulation, the mass matrix is diagonal to a good approximation. The element  $M_{11}^{-1}$  turns out to be identical to  $1/M_e$  of Eq. (41); neglecting terms of order  $J_\pm/J$ , the other diagonal elements are

$$M^{-1} = \frac{1}{2} \frac{\Omega - \omega_+}{M_e} \quad M^{-1} = \frac{1}{2} \frac{\Omega - \omega_-}{M_e}$$

The matrix  $K$  is

$$K = \text{diag}(2\beta\sqrt{J}, 2\alpha\sqrt{JJ_+}, 2\alpha\sqrt{JJ_+}). \quad (63)$$

Using the values obtained before for the equilibrium, we determined the eigenvalues numerically from the full matrix. The three oscillation frequencies are

The first of these corresponds to the large oscillations observed in the simulations. The eigenvector of this mode corresponds primarily to the  $S$  degree of freedom that describes oscillations

the frequency of oscillations of the asymptotic state with the calculated value. We have no evidence

modes; however, it might be possible to determine these through careful simulation.

Therefore, we conclude that the chaotic sea model

much better than the single particle calculation

### 5. Conclusions

We have studied the long-time dynamics of the

electrostatic interaction of many particles with a plasma wave. The wave arises from an instability (the beam-plasma instability) of the initial state corresponding to a cold beam of particles. In the simulations, the asymptotic state corresponds to a periodically oscillating wave amplitude together with a trapped clump of particles. About 42% of the particles are trapped by an approximate invariant surface within the oscillating wave, while the remaining particles move chaotically—becoming successively trapped and detrapped.

We modelled this motion by a four degree-of-

chaotic sea, and one to the wave. This model quantitatively captures the asymptotic state of the effectively infinite degree-of-freedom system.

One would like to speculate that there are other physical systems for which the effect of self-consistency would be similar. For example in the case of galactic dynamics, the self-consistent propagation of a density wave would

there could be a set of stars which are coherently interacting with the wave.

As usual, a number of open questions remain:

The first is, how can we test the asymptotic state

the QWM model. What is the “basin” of initial

for example, consider a warm beam initial state

discussed at the end of Section 3.1.

– Is there a way of self-consistently calculating the quantities  $N_{tr}$  and  $n_{tr}$  which we took from



particles at rest at the maximum of the potential.  
Is there a periodic state of the many particle system near this equilibrium?

In the QVM model, only a single Fourier harmonic is considered.

the effect of adding additional harmonics?

The chaotic sea Hamiltonian should exhibit the full array of possible Hamiltonian motions. For example there should be fixed points, no

manifolds leading to homoclinic phenomena.

states that correspond to these motions?

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### References

[1] M. G. Velikhov, *Phys. Fluids* 14 (1971) 1204-1212.  
[2] N. G. Moshchilov, I. N. Onishchuk, V. D. Shabat and V. I. Shevchenko, On nonlinear theory of instability of a monoenergetic electron beam, *Plasma Phys.* 14 (1972) 591-600.

nonlinear beam-plasma interaction, *Phys. Fluids* 21 (1978) 653-663.

[6] G.R. Smith and N.R. Pereira, Phase-locked particle motion in a large amplitude plasma wave, *Phys. Fluids*

nonlinear Langmuir waves, *Phys. Fluids* 24 (1981) 260-267.

[7] J.R. Cary, et al., Resonance and near resonance near diffusion far from the chaotic threshold, *Physical Review Letters* 65 (1990) 3132-3135.

three-wave interaction: integrable case of this system

[10] J.R. Cary, I. Doxas, D.F. Escande and A.D. Verga, Enhancement of the velocity diffusion in longitudinal plasma turbulence, *Phys. Fluids B* 4 (1992) 2062.

[11] P.J. Morrison, The Maxwell-Vlasov equations as a continuous Hamiltonian system, *Phys. Lett. A80* (1980) 383-386.

[12] P.J. Morrison, Hamiltonian field descriptions of two

Princeton Plasma Physics Laboratory Report PPPL-1783 (1981), available as American Institute of Physics

Publication Service, 335 East 45th Street, New York, NY 10017).

[13] A.N. Kaufman and R.L. Dewar, Canonical derivation of the Vlasov-Coulomb noncanonical Poisson bracket,

[14] P.J. Morrison and D. Pfirsch, Dielectric energy versus plasma energy, and Hamiltonian action-angle variables for the Vlasov equation, *Phys. Fluids B4* (1992) 3038-3057.

[15] R. Shadwick and P.J. Morrison, On neutral plasma

Systems: a Rerprint Selection (Adam Hilger, 1987)

[16] C. Kuery, Ph. D. thesis Nonlinear instability and chaos

[17] T.M. O'Neil, Collisionless damping of nonlinear plasma oscillations, *Phys. Fluids* 8 (1965) 2255-2262.

[18] R.O. Davidson, *Methods of Nonlinear Plasma Theory* (Academic, New York, 1972), pp. 45-47.