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Abstract

A sequence of n independent and identically distributed random variables X_1, \dots, X_n is considered. The partial sums $S_k = X_1 + \dots + X_k$ are defined for $k = 1, \dots, n$. The maximum of the partial sums $M_n = \max_{1 \leq k \leq n} S_k$ is studied. The asymptotic behavior of the distribution of M_n is investigated. The limiting distribution of M_n is shown to be the Gumbel distribution. The asymptotic expansion of the distribution function of M_n is obtained. The asymptotic behavior of the moments of M_n is also studied. The asymptotic behavior of the probability of a large deviation of M_n is investigated. The asymptotic behavior of the probability of a large deviation of M_n is shown to be the same as the asymptotic behavior of the probability of a large deviation of S_n . The asymptotic behavior of the probability of a large deviation of M_n is shown to be the same as the asymptotic behavior of the probability of a large deviation of S_n .

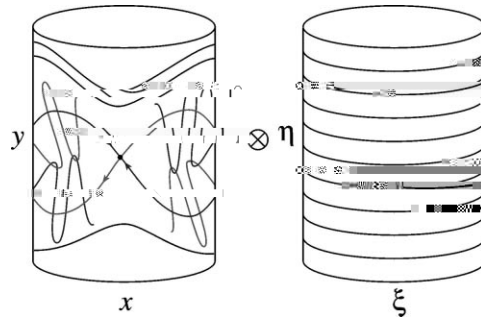
MSC: 37.40; 37.40; 37.50

Keywords: Random walk; Maximum of partial sums; Large deviations

1. Introduction

A sequence of n independent and identically distributed random variables X_1, \dots, X_n is considered. The partial sums $S_k = X_1 + \dots + X_k$ are defined for $k = 1, \dots, n$. The maximum of the partial sums $M_n = \max_{1 \leq k \leq n} S_k$ is studied. The asymptotic behavior of the distribution of M_n is investigated. The limiting distribution of M_n is shown to be the Gumbel distribution. The asymptotic expansion of the distribution function of M_n is obtained. The asymptotic behavior of the moments of M_n is also studied. The asymptotic behavior of the probability of a large deviation of M_n is investigated. The asymptotic behavior of the probability of a large deviation of M_n is shown to be the same as the asymptotic behavior of the probability of a large deviation of S_n . The asymptotic behavior of the probability of a large deviation of M_n is shown to be the same as the asymptotic behavior of the probability of a large deviation of S_n .

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Let $\mathbf{A} = \mathbf{A}(x, y)$ be a vector field on $T^2 \times \mathbb{R}^2$. (1) $\mathbf{w} \cdot h = 0$... (x, y) ...

... $\mathbf{A} = \mathbf{A}(x, y)$... $T^2 \times \mathbb{R}^2$... $x' = x + y'$, $y' = y - k(1 + h \dots) x$, $\dots = -kh(\dots)$

$(x, y) \mapsto (x', y'),$
 $x' = x + \frac{1}{2}y, \quad y' = y + V(x).$
(4)

$n = 1, \dots, V(x) = k \dots (x),$

3. A t - t a b - t

$W(x) = \int_t V(x_t),$
(3)

$V(x_t) = 0,$
(6)

$(V) \equiv \{c : V(c) = 0, \| \nabla^2 V(c) \| \geq b > 0\}.$
(5)

$B = (\mathbb{R}^n)^{\mathbb{Z}}$
 $X \in B,$
 $A_t(X) \equiv -T_2(x_{t-1}, x_t) - T_1(x_t, x_{t-})$

4. Comparison of t_{max}

As a first step in comparing the two models, we compare the time to reach maximum concentration, t_{max} , for the two models. For the two models, we assume that the initial concentration is zero, $C(x, 0) = 0$, and the initial velocity is zero, $v(x, 0) = 0$. For the two models, we assume that the initial concentration is zero, $C(x, 0) = 0$, and the initial velocity is zero, $v(x, 0) = 0$. For the two models, we assume that the initial concentration is zero, $C(x, 0) = 0$, and the initial velocity is zero, $v(x, 0) = 0$.

that C is not identically zero. Then given any $a < b$, there is a nonzero measure of initial states (x_0, x'_0) and a sequence $c_t \in (V)_+ \cup (V)_-$ such that the solution of (14) has momenta, $x_t = T_2(x_{t-1}, x'_t)$ satisfying $a < x_t < b$ and $T > b$ for some time T .

P ... (14) ... $c_- \in (V)_-, c_+ \in (V)_+, x_t = c_{\pm} \dots (C(t)) = \pm 1$.

$$\tilde{L}(x, x') = T(x, x') + W(x) + \tilde{C}(x),$$

... $\tilde{C} = V(c_{\pm}(x)) \quad C(x) \geq 0$.

$$\tilde{C}(x+2) - \tilde{C}(x) > 0, \quad F = \tilde{L}(x+2, x'+2) - \tilde{L}(x, x')$$

... $\tilde{C}(x+2) - \tilde{C}(x) > 0, \quad F = \tilde{L}(x+2, x'+2) - \tilde{L}(x, x')$... \square

4.2. Standard example

$$L(x, x', t, t') = \frac{1}{2} (x' - x)^2 + \frac{1}{2} (t' - t)^2 + k \dots x(1 + h \dots), \tag{15}$$

... $k > 0, h > 0$... $\mathbf{A} \dots (1), \dots (15)$

$t \rightarrow \dots$
 $^* = 2 - m$
 $(0, 2 - m) \dots (2 - m)$
 (16)

$^* = (2m + 1)$
 $(0, ^*) \mapsto (^*, ^*)$
 (16)

g
 g^*
 g^*
 g
 (16)

$T^2 = \{(,) : 0 < , < 2\}$
 $^* = (t_{+1} - t)^* = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t=0}^{T-1} (U'(t))$

$\langle ^* \rangle = \frac{1}{4 - 2} \int_{T^2} U'() = \frac{F}{2}$

(16)

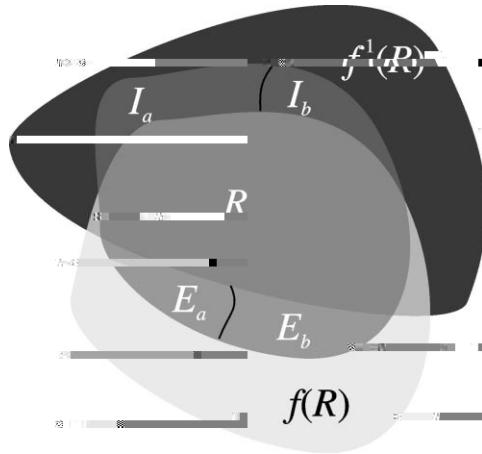


Fig. 4. The set R and its image $f(R)$.

where

$$E = \{z \in R : f(z) \notin R\} = R \setminus f^{-1}(R).$$

Since f is a contraction, $\mu(f^{-1}(R)) < \mu(R)$. The set E is the part of R that does not map to R under f .

$$\mu(E) = \mu(R \setminus f^{-1}(R)) = \mu(R) - \mu(R \cap f^{-1}(R)) = \mu(R) - \mu(f(R) \cap R) = \mu(R \setminus f(R)) = \mu(I). \quad (17)$$

Let S^t be the set of points in R that stay in R for t iterations of f . Then

$$S^0 = I, \quad S^t = f(S^{t-1}) \cap R = f(S^{t-1} \setminus E).$$

Since $S^t \subset R$, $\mu(S^t) < \mu(R)$. Also, $\mu(S^t) \rightarrow 0$ as $t \rightarrow \infty$. The set E is the part of R that does not map to R under f .

$$\mu(p(I_a) \cap E_b) = \mu(p(I_a)) - \mu(p(I_a) \cap E_a) \geq \mu(I_a) - \mu(E_a). \quad \square$$

A f_t R

$$I_t = R \setminus f_t(R), \quad E_t = R \setminus f_t^{-1}(R).$$

(17) f_t R k t S_k^t

$$S_k^k = I_{k-1}, \quad S_k^{t+1} = f_t(S_k^t \setminus E_t).$$

A S_k^t t $k \leq t$ R

$$\mu(S_k^t) < \mu(R) \quad (18)$$

L **a 4.** Let f_t be a sequence of measure-preserving homeomorphisms, and R a measurable set with incoming sets I_t and exit sets E_t .

5.2. Maps of the cylinder

Let $f: C \rightarrow C$ be a map of the cylinder C with net flux F . Let C be identified with \mathbb{R}/\mathbb{Z} and f with $f(x) = x + F + g(x)$, where g is a periodic function with period 1.

$$F = \int_C f'(x) dx.$$

Let A be an annulus bounded by the circles $\{y = a\}$ and $\{y = b\}$ where $a < b$. Let T and B be subsets of C such that $U \subset T$ and $D \subset B$.

$$U = \{z \in T : f^{-1}(z) \in B\}.$$

Let $D \subset B$ and T be subsets of C such that $D = \{z \in B : f^{-1}(z) \in T\}$.

$$\mu(U) - \mu(D) = F.$$

Proposition 5. Suppose that f_t is a sequence of area and end-preserving homeomorphisms of the cylinder, and that the net flux $F_t \geq \epsilon > 0$. Let A denote the annulus bounded by the circles $\{y = a\}$ and $\{y = b\}$ where $a < b$. Then, there is a set of positive measure of orbits that cross the annulus.

Proof. Let $U_t(a)$ and $D_t(a)$ be the sets U and D for f_t with $T = \{y = a\}$ and $B = \{y = b\}$. Let $U_t(b)$ and $D_t(b)$ be the sets U and D for f_t with $T = \{y = b\}$ and $B = \{y = a\}$. Then $E_t = f_t^{-1}(D_t(a)) \cup f_t^{-1}(U_t(b))$ and $F_t = \mu(U_t(b)) - \mu(D_t(a)) \geq \epsilon > 0$. \square

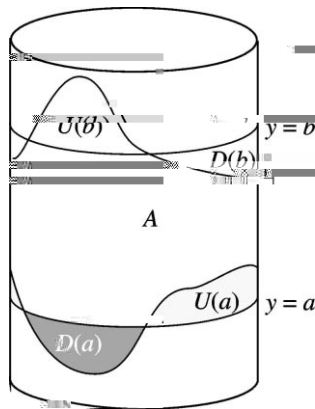


Fig. 5. A diagram illustrating the cylinder map and the annulus A.

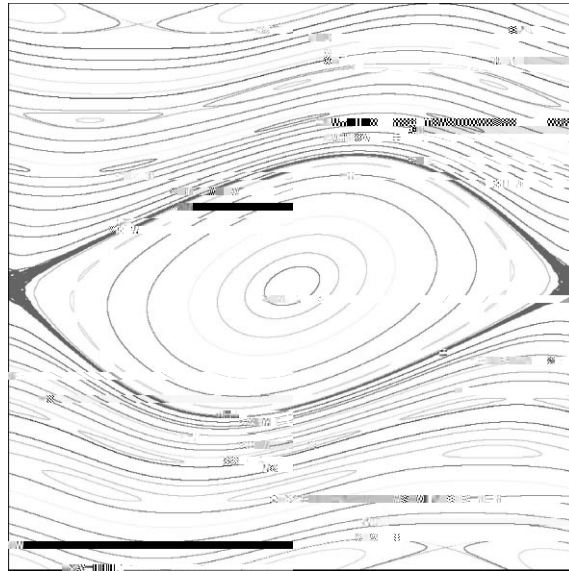


Fig. 6. Phase portrait of the standard map with net flux, $k = 0.5$, $\tau = 0.5$.

5.3. Standard map with net flux

Consider the standard map with net flux, $(x, y) \in (0, 2) \times \mathbb{R}$, $\tau \in (0, 1)$, $F = V(2 - y) - V(y)$, $V(y) = \frac{1}{2}y^2$, $\tau = 0.5$, $k = 0.5$. The map is given by

$$x' = x + y', \quad y' = y - k(x) + \frac{1}{2}F.$$

For $F = 0$, $k < k_{cr} \approx 0.971635406$, the map is integrable. For $k = 0.5$, $F \neq 0$, the map is non-integrable. The phase portrait shows a central vortex surrounded by a ring of points. The trajectories are more densely packed in some regions and more spread out in others, suggesting a complex underlying structure like a quasiperiodic ring or a chaotic attractor. The overall shape is roughly diamond-like or square with rounded corners.

$$x = \frac{F}{2k}$$

For $F = 2k$, the map is integrable.

6. Periodic orbit

$$z_{t-1} = (x_{t-1}, x_t, t-1, t)^T, \quad (12)$$

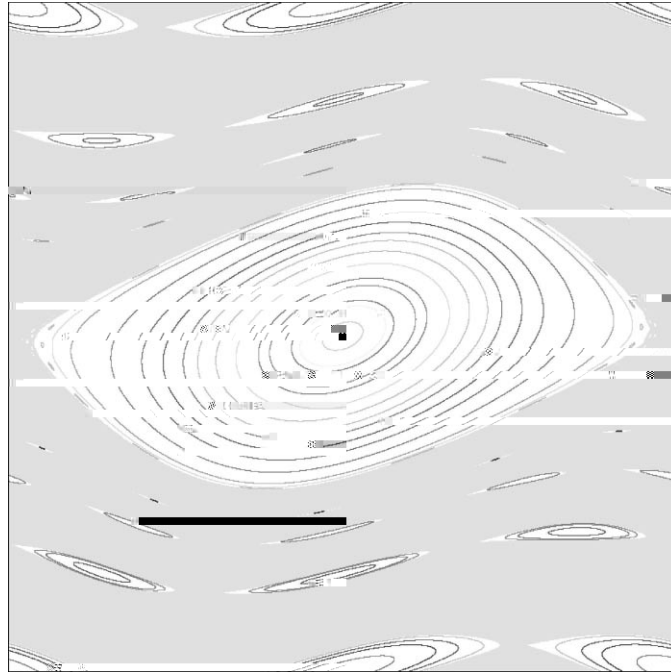


Fig. 7. Contour plot of the potential function $V(x)$ for $k = 0.5$, $F = 4^{-2}/1000$. The contours are centered at $x = 0$.

$$z_t = (x_t, t) = \left(-x_{t-1} + 2x_t + \frac{1}{t} V(x_t) + C(t), -t_{-1} + 2t + W(t) + V(x_t) - C(t) \right) \quad (19)$$

where $C(t) = 0$, $W(t) = 0$, $x_t = c_t \in \text{int}(V)$, $V(c_t) = 0$, $V'(c_t) > 0$, $V''(c_t) < 0$.

Lemma 6. Suppose that z_t , given by (19), is a C^2 map of \mathbb{T}^4 , such that $1 + C(t) \geq a > 0$. Then, for any sequence $\{c_0, c_1, \dots\}$ with $c_t \in \text{int}(V) \cap A$, any initial condition (x_0, t_0) , and any $\epsilon > 0$, there exists an orbit $z_t = (x_t, t_t)$, $t_t \geq t_0$ of z_t such that

$$|x_t - c_t| \leq \epsilon \quad t \geq 0,$$

provided

$$0 \leq \epsilon < \epsilon_0 = \frac{1}{(4 + a)}, \quad (20)$$

where $(\epsilon, b) \equiv \sup_{t \geq 0} |V(c_t \pm \epsilon)|$.

(1/2) $V(c_{t+1}^+)$ $> (4 + a)$ \dots $1 + C(\dots) \geq \dots$ $> 4 + a$ (20).
 $S = \mathbb{R}^2 \times (0, 1) \cap W_0$. U_t W_t $t \geq 1$. B_{t+1} W_{t+1} .
 T B S W_t T B . \square

$2, \dots$ $(0, 1)$ $\{c_t\} \in \dots(V)$.
 $Z_t = (c_t, c_{t+1}, t, t+1)$ $\{Z_t\}$ $\{Z_t\}$

T **7.** Suppose that \dots satisfies the hypotheses of Lemma 6. Let $Z_t = (c_t, c_{t+1}, t, t+1)$ be an orbit of \dots with $c_t \in \dots(V)$. Then for any $T \geq 0$ and $\dots > 0$, there is a $\dots > 0$ such that for all $\dots < \dots$ in (20.5716 0 0 7.5716 439. 1

For $|t| \leq r^t$, $r > 1$, $r^2 - wr - 1 = 0$, $w = \frac{1}{2}x(2 + |W(x)|)$.
 $|t| \leq \frac{1}{2}M^2 r^{2t}$.

For $T \leq t \leq T^2$, $W = 0$, $t \leq T^2$. \square

R a $C(\cdot)$

6.1. Standard example, continued

(15), $V(x) = k|x|$,
 $C(\cdot) = h$, $h < 1$.
 $a = 2$, 6
 $\leq 0 = \frac{k(1-h)}{4+2}$.

$M = kh$, $W = 1$, $DB)DB$, $b B Db$, $b b$, $ET bDD B$

... $h < 1$... ≤ 0 ... 9.

7. Conclusion

... 7 ... 6,9,20 ... 19 ... 16 ... (x, y) ... 17.

Appendix A

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